AIEM

AVANCES DE INVESTIGACIÓN EN EDUCACIÓN MATEMÁTICA

Revista de la Sociedad Española de Investigación en Educación Matemática

ISSN-e 2254-4313 | (2024) 26, 85-103 | aiem.es

SEIEM

seiem.es

Utilizing Decision-making Task: Students' Mathematical Justification in Collaborative Problem Solving

Utilización de una tarea de toma de decisiones: justificación matemática de los estudiantes en la resolución colaborativa de problemas

Namirah Fatmanissa @, Tatag Yuli Eko Siswono @, Agung Lukito @, Ismail @

State University of Surabaya (Indonesia)

Abstract ∞ Many studies reported the importance of mathematical justification in collaborative problem-solving (CPS). However, not all tasks could stimulate mathematical justification in CPS. This study explores the potential of a decision-making task in facilitating mathematical justification in CPS of a derivative topic. Two groups of 12 graders in Bandung, Indonesia solved a task. The group works were observed, recorded, and the written works were collected. The findings showed that the task encouraged the groups to focus on justifying mathematical claims. Both groups successfully solved the task, yet different mathematical justifications were observed. We discussed the possible roles of the task difficulty and groups' mathematics ability in promoting mathematical justifications. Checking the effectiveness of the task on a larger sample was recommended for further studies.

Keywords ∞ Collaboration; Derivative; Justification; Problem-solving; Decision-making task

Resumen ∞ Muchos estudios informaron la importancia de la justificación matemática en la resolución colaborativa de problemas (CPS). Sin embargo, no todas las tareas podrían estimular la justificación matemática en CPS. Este estudio explora el potencial de una tarea de toma de decisiones para facilitar la justificación matemática en CPS de un tema derivado. Dos grupos de alumnos de 12º grado en Bandung, Indonesia, resuelven una tarea. Los trabajos grupales fueron observados, grabados y recopilados los trabajos escritos. Los hallazgos mostraron que la tarea animó a los grupos a centrarse en justificar afirmaciones matemáticas. Ambos grupos resolvieron con éxito la tarea, aunque se observaron diferentes justificaciones matemáticas. Discutimos los posibles roles de la dificultad de la tarea y la habilidad matemática de los grupos en la promoción de justificaciones matemáticas. Se recomendó para estudios posteriores comprobar la eficacia de la tarea en una muestra más grande.

Palabras clave ∞ Colaboración; Derivación; Justificación; Resolución de problemas; Tarea de toma de decisiones

Fatmanissa, N., Yuli Eko Siswono, T., Lukito, A., & Ismail (2024). Utilizing Decision-making Task: Students' Mathematical Justification in Collaborative Problem Solving. *AIEM - Avances de investigación en educación matemática, 26,* 85-103. https://doi.org/10.35763/aiem26.5341

1. INTRODUCTION

Mathematical justification is an activity of providing arguments that support or refute a mathematical claim by using prior mathematical knowledge and meeting the requirements of the community (de Villiers, 1990; Staples et al., 2012). Cirillo et al. (2016) stated that it is the essence of reasoning activity in problem-solving. The habit of giving reasons for the truth of a claim or steps taken in solving a problem encourages good reasoning skills (Brodie, 2010b; NCTM, 2000). By justifying a claim, students try to find reasons for their claims and do not simply accept the ideas they receive.

Several studies stated that mathematical justification determines the success of collaborative problem-solving (CPS) (Chiu, 2008; Díez-Palomar et al., 2021). Chiu (2008) analyzed problem-solving in groups of high school students and found that justification had the most significant effect on the success of a group. In addition, Díez-Palomar et al. (2021) found that student interactions dominated by mathematical justification were closely related to whether or not a group's answers were correct because students checked the validity of claims and made them the basis of their answers. Thus, the group's success was related to mathematical justification, which provides space to question why their steps can be conducted and are legitimate to reach answers.

Despite its importance, some studies reported that mathematical justification was neither practiced nor demanded by students in CPS (Hamidy & Suryaningtyas, 2016; Stylianou & Blanton, 2002). Students tended to go through the solving process to get an answer, not to argue why the answer was valid. Besides, teachers found it hard to encourage mathematical justification in the classroom by facilitating them into it or creating stimulating tasks (Brodie, 2010a). The role was challenging as it required teachers to create a situation where students questioned themselves and found a way to convince others mathematically. It led to the need to investigate a task that could accommodate such activity.

Cobb et al. (1992) stated that mathematical justification happened when there was a *situation for justification*, a situation where a claim was questioned and requested for validation. Not any task can stimulate such a situation. Chua (2017) introduced one such task as a decision-making task. In decision-making, students would be asked to decide whether an answer or statement was correct and explain why. It could be inferred that the task might stimulate the situation for justification and, thus, the mathematical justification itself. This study explores the strengths and limitations of decision-making task in facilitating mathematical justification, especially in the context of CPS. Through this, we explored how students interacted with each other in the problem-solving process and identified mathematical justification made by them.

2. Theoretical Framework

2.1. Mathematical justification

Mathematical justification can be interpreted as an activity to refute or defend a claim by using statements or accepted mathematical reasoning or as an argument that is the product of the activity (Staples et al., 2012). Similarly, Yackel and Cobb (1996) defined mathematical justification as an argument or agreement on an acceptable mathematical explanation for a mathematical method to be used. Mathematical justification involves "acceptance" of the individuals based on the agreement that a mathematical claim is reasonable (Bieda et al., 2022). Thus, mathematical justification also considers the social context in which mathematical activities are carried out (Simon & Blume, 1996; Staples & Conner, 2022).

Mathematical justification is a cognitive and a social activity. In carrying out mathematical justification, besides using cognition in arguing to support (or refute) a claim, one also requires effort to convince others in that activity (Sowder & Harel, 1998). Due to the involvement of social considerations, there are times when mathematical justification is not necessarily logically complete (Jaffe, 1997; Kilpatrick et al., 2001). For example, a study by Sowder and Harel (1998) asked students to check whether a rhombus would be formed if the midpoints of the sides of an isosceles trapezoid were connected. Most students answered yes and justified by drawing an isosceles trapezoid, making the midpoints of the sides, and showing that the points were connected to form a rhombus. The student in this study chose drawing to convince others of his mathematical claims although logically incomplete.

The decision on what kind of argument is considered good mathematical justification is based on how convincing (plausibility) the argument is (Walton, 2001). In this regard, several studies identified levels of mathematical justification. Simon and Blume (1996) divided students' mathematical justification into five levels, i.e., level 0 (without justification), level 1 (appeal to external authority), level 2 (empirical demonstration), level 3 (deductive justification expressed through examples), and level 4 (deductive justification independent of example). Vale et al. (2016) identified similar levels, while Sowder and Harel (1998) and Carpenter et al. (2005) considered them as types but explained that one type was better than those mentioned earlier.

We acknowledged that the choice of any classification scheme would emphasize some aspects of justifications and, at the same time, de-emphasize others. In this study, a statement by Ellis et al. below represented our thoughtswhen choosing the classification scheme.

> When one seeks to examine the co-construction and enactment of justification norms and practices ..., which mathematical ideas were justified, and the extent to which members of the classroom community were convinced by the justifications, a broader accounting of the nature of justifications and justification activity is appropriate. (2022, p. 291)

We resorted to having five levels of mathematical justification, similar to Vale et al. (2016), to help us describe the variety of mathematical justifications while utilizing the decision-making task (Table 1). We think classifying the justifications into five levels would help us see the broader quality of efforts (or absence of effort) made by students to justify their claims.

Table 1. Coding scheme

Justification code	Description
P	Trigger of a situation for justification
L0	No justification
L1	Appeal to authority
L2	Empirical or perceptual demonstration
L3	Symbolic example without generalization
L4	Symbolic example with generalization
Interaction code	**Description**
Int.	Students' thinking is affected by their interactions within the group.
Syn.	Students expect others to wait for a talk or idea and process it as soon as being received.
Neg.	One student does not force his idea on his peers within the group.
Participant code	**Description**
Ti	The i-th* participant with a high math ability
Si	The i-th* participant with average math ability
Ri	The i-th* participant with low math ability

* Order was made to differentiate participants of the same mathematics ability, not for ranking purposes

At level 0, students accepted a claim without trying to justify the truth of the claim. At level 1, students state that a claim is valid because it comes from a more "authoritative" source outside themselves, such as a textbook or teacher. For example, a student does a particular strategy because his teacher suggested using that strategy beforehand. It is interesting to note that in these two first levels, it is probably not suitable to call them an act of mathematically justifying claims, as no mathematical argument (level 1) or even no argument (level 0) was presented. However, we consider these two levels as important as other levels in terms of helping us see and understand what students thought as valid claims or even whether they think they need to justify them in the first place.

At level 2, students provide justification based on empirical demonstrations, i.e., by what they see and show it to others by demonstrating it. For example, students support the claim that a picture of a quadrilateral is a model of a square because the sides are the same length. He may also convince others by measuring the length of the sides, which are indeed the same length, with some measuring instruments. This student justifies by referring to what he sees and experiences when interacting with the picture of the quadrilateral.

At level 3, students begin to use symbolically expressed examples that have not been generalized. Sowder and Harel (1998) provided an example at this level when students were asked if $n^2 - 79n + 1601$ is prime for any value of n. In this example, a student justified the claim by giving examples of the values $n^2 - 79n + 1601$ for several values of n and concluded that the calculation always gave prime numbers without generalizing to all possible values of n. At this level, the student confirmed his claims to others by using examples, and because he did not use generalizations, he did not see that for $n = 80, n^2 - 79n + 1601$ is not prime. At level 4, students make justifications based on generalizations that do not rely on examples, or they use definitions, theorems, or rules of mathematical logic. Vale et al. (2016) stated that this level was the most difficult for students to find because they did not feel that this justification was necessary to have their claims accepted by other students.

2.2. Interactions and mathematical justification in collaborative problem solving

In this section, we try to elaborate on literatures in understanding students' interactions and mathematical justification in the context of CPS. Students will go through stages of solving problems collaboratively, i.e., analyzing, planning, implementing, and evaluating (Rott et al., 2021) while interacting. It is undeniable that students might have different quality of interaction during CPS.

Dillenbourg (1999) stated three criteria to see whether students interact collaboratively, i.e., interactivity, synchronicity, and negotiability. Interactivity refers to how interactions affect students' thinking within a group. By this, interactivity goes beyond how often students took turns or talked to each other but how they become responsive to others' thinking. Another criterion is 'doing something together' or synchronicity. It refers to how each student expects others to wait for a talk or idea and will process that idea as soon as being received. The synchronicity also shows how, in CPS, collaboration happens when giving, listening, and responding to ideas synchronously. Synchronicity shows that students do not work in parallel or through a division of labor but are continuously alert to others' efforts throughout the CPS process (Jermann, 2004). Negotiability refers to how one student cannot force his idea on his peers but must construct a common understanding. It shows that the interactions are made to reach a common understanding of each student within the group. Negotiability discourages each student from understanding only a portion of the effort or goal, thus resulting in a division of understanding (Lai, 2011). To understand the use of the decision-making task in this study, we resorted to identifying the three criteria by Dillenbourg to help us describe the interactions that happened during CPS (Table 1).

Student statements may be questioned during discussions in the CPS stages. A co-constructed justification is formed when the group agrees upon students' mathematical justification (Mueller, 2009; Yackel, 2004). A similar concept was proposed by Tatsis and Koleza (2008) and Yackel and Cobb (2001) as the norm of mathematical justification, regularities in collective activity when a mathematical method or claim needs to be supported by a reason. In other words, mathematical justification in the CPS situation can be seen as a mathematical justification built jointly by a group or as a regularity of mathematical justification activities when a group solves a problem. Co-constructed mathematical justification or norms of mathematical justification determine whether the ongoing CPS process can proceed to the next stage or whether it is necessary to return to the previous stage (McClain & Cobb, 2001; Partanen & Kaasila, 2015; Tatsis, 2007).

2.3. Task stimulating mathematical justification

Mathematical justification can be observed by providing tasks that trigger claims and arguments supporting those claims. It is important to select tasks or problems that support students' emergence of mathematical justification (Heid et al., 2002) because students will be provoked to convince others of their mathematical arguments. Chua (2017) described a decision-making task as one of the tasks stimulating justification. In this task, students are asked to decide whether a statement is correct and explain why. An example of a decision-making task is given in Figure 1. In this task, students had to decide whether 207 is a term in the given sequence and argue to support that decision.

Figure 1. Example of decision-making task

The first four terms of a sequence are 5, 9, 13 and 17. Explain whether 207 is a term in the sequence.

Source: Chua, 2017, p. 120.

Decision-making tasks offer choices for a mathematical decision and forced students to focus on supporting or refuting the claim stated within. The task was used in several studies (Chua, 2016; Küchemann & Hoyles, 2006) investigating individual problem-solving and showed its strengths in stimulating mathematical justification. The exploration of such a task being given in a CPS situation would bring insights into its strengths in a collective setting.

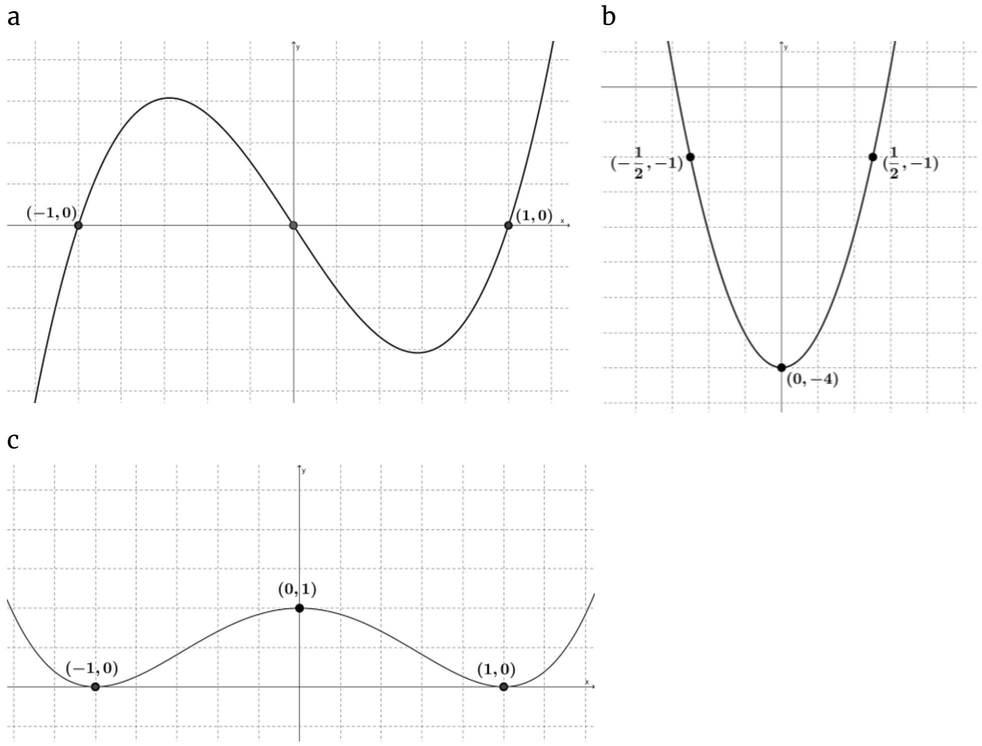
3. Methods

A task of derivative (Figure 2) was given to two groups of three 12th-grade students. They were chosen based on their mathematics ability, determined by their accumulated report scores in 10 and 11 grades. In this study, we focused on exploring the mathematical justifications of groups with varied abilities which we believed were two of the typical group sets in ordinary classrooms. Further, several studies found that students of different mathematics abilities tended to give different justifications (e.g., Chua, 2016; Staples & Conner, 2022; Whitacre et al., 2017). On that point, the group setting might enrich our understanding of how students justified their thinking to their peers whose thinking might differ. Group 1 consisted of students with high (T1), average (S1), and low (R1) mathematics abilities. Group 2 consisted of two students with a high ability (T2 and T3) and one with an average ability (S2).

The task was designed following Chua's description of a decision-making task (2017). There are two considerations in constructing the task: it contained an explicit statement to be validated and required a derivative concept to solve. The three functions in the task were presented graphically to allow for different justifications upon functions equations, knowing that some students might rely only on perceptual (in this case, graphical) demonstration to justify claims. Two mathematical education experts and the participants' mathematics teacher consulted the construction to ensure its readability and clarity. The test was written and administered in Indonesian.

Figure 2. The task for participants

Andi is given three graphs as follows. Among the three graphs, he is asked to determine which one is the graph for function f, f', and f''. Andi stated that a is the graph of f, b is the graph of f', and c is the graph of f''. Is Andi's statement true? Justify your answer.



In the task, students were expected to determine the truth of Andi's statement by checking if a is the graph of f, b is the graph of f', and c is the graph of f''. The expected answer is that Andi's statement is wrong, but coming to this conclusion might be due to various strategies and justifications. One possibility is that students want to determine the degree of each polynomial function based on the graphs. For example, they might successfully determine that a is a polynomial function of degree three, b is a quadratic function, and c is a polynomial function of degree four by considering the number of stationary points and shape of each function. By coming to this information, they might realize that Andi's statement that c is the derivative of b is incorrect because it is impossible to have a four-degree polynomial function as the derivative of a quadratic function. The way they come to this conclusion is justified by how they perceptually see the stationary points and shape of the graph.

In another possibility, students might not be satisfied by only knowing the degree of the function. They might go to higher justification by determining the exact equation of each function to defend the relationship of the functions. For example, they might have determined that $b(x) = 12x^2 - 4$ from how the graph is stationary at $(0, -4)$ and passes through $\left(-\frac{1}{2}, -1\right)$ and $(\frac{1}{2}, -1)$. They might also use the derivative rule to determine the derivative of b and result in $b' = 24x$, which does not belong to any function in the task. Another possibility is that students choose to determine what function has a derivative of $12x^2 - 4$ and conjectured that the function is a based on their understanding of the derivative rule.

The task topic and plan were announced to students a week before the administration. Students could use a non-graphical calculator and were instructed to discuss or ask their teammates whenever they were uncertain. The task was discussed for an hour, and each group was expected to write their final answer. The researcher provided the paper, and each group was given only one page to ensure information sharing and joint effort. Conversation in each group was recorded, and all paper works were collected. Discussion upon work was transcribed and coded based on the scheme in Table 1.

4. Findings and Discussions

In this section, we explored the use of decision-making tasks by looking at the justifications during the solving processes and interactions of the two groups. Both groups showed interactions among teammates, and various mathematical justifications were observed. However, different interactions and processes were evident between groups in solving the same task.

4.1. Group 1 (T1, S1, and R1)

Students in this group began the CPS process in the analysis phase by identifying the shape of each graph. Then, students did not explicitly discuss the plans they would do but immediately entered the implementation phase. First, T1 recognized graph b as a graph of a quadratic function and was agreed by the other two students. He convinced the other two students by stating that he had seen their teacher explain the shape of the graph of a polynomial function of degree three, whose shape was like graph a. In this case, he performed an appeal to authority type of mathematical justification. Students S1 and R1 agreed because they remembered the same thing.

T1 concluded that function b is the derivative of function a. A mathematical justification was carried out by students S1 and T1 when R1 questioned the conclusion that quadratic functions are derivatives of cubic polynomial functions.

Students S1 and T1 provided mathematical justification by explaining an example, as presented in the transcript below.

S1: This one (graph a) could be $x^3 - x^2 + 3x - 12$, something like that.

T1: Right.

S1: Then we differentiate into $3x^2$ ... (*pause*, *looking at R*)

R1: Yes, just try it (*laughing*)

S1: So, $3x^2 - 2x + 3$, right?

T1: See, that x^2 is quadratic.

R1 agreed upon the mathematical justification presented by S1 and T1. They used an example to support the claim that the derivative of a cubic polynomial function was quadratic. Furthermore, the results of their discussion led to the fact that it was impossible for graph c to be the derivative of graph b because if so, graph c would be a linear function. R1 then asked, "Is it possible to find the equation of function c by observing the ups and downs of the graph?" R1's question led to S1's idea to study the graphs of polynomial functions of even degrees. R1 gave an example that the function $y = x^4$ shaped like the letter "U". S1 then convinced students R1 and T1 that the graph of c is a polynomial function of degree four because the shape of the graph of c resembles the shape of the graph of the function $y = x^4$. S1 demonstrated a mathematical justification of a perceptual demonstration. They closed this task by concluding that Andi's statement was wrong and curve c was a graph of a function f, curve a is f', and curve b is f''. This group did not evaluate their solution and wrote their final work which is translated into "The equation of graph c has the greatest power, which is 4 (x^4), if being differentiated, the (resulting) equation has the highest power of 3 (x^3) and graph a is the graph whose equation contains x^3. Finally, if the equation of graph a is differentiated, it will be quadratic, and graph b is quadratic.

From the CPS process in Group 1, all criteria of collaborative interaction were observed. While R asked for an explanation of a certain claim, this request affected S1 and T1's thinking and triggered them to give arguments to support the claim. Interactivity was also seen in how R1 accepted S1 and T1's justification. The group's synchronicity was shown by how students took turns in talking and being involved in the CPS process together. The negotiability was apparent that despite most justification and claims given by T1 and triggers and doubts coming from S1 dan R1, it was evident that no student forced his idea on others.

In summary, during the discussions, five claims emerged, concluding that Andi's statement in the task was wrong, and efforts to justify them were observed (Table 2). In completing this task, students carried out mathematical justification in several ways, i.e., using the teacher's explanation to support their claim, using an example to show the truth of a claim, and using what they saw from the shape of the function graph. Students T1 and S1 provided mathematical justification to convince students R1 or to form an agreement for the group's conclusions. R1 triggered the emergence of mathematical justification through his questions, even though he did not provide mathematical justification. Students T1 and S1 provided different mathematical justifications.

Table 2. Claims and justifications made by Group 1

Claim (by order of appearance)		Mathematical justification	Students' utterances/ activity
K1	a is a cubic polynomial function	Appeal to authority	"I had seen the teacher explaining the shape of it."
K2	b is a quadratic function	Appeal to authority	"Our teacher had shown us this shape before."
K3	b is the derivative of a	Symbolic example without generalization	Giving an example of a cubic polynomial function and differentiating it into a quadratic function
K4	c is the polynomial function of degree 4	Empirical or perceptual demonstration	Demonstrating that the function $y = x^4$ shaped like the letter "U"
K5	b is the derivative of a, and a is the derivative of c	Symbolic example without generalization	Giving an example that the derivative of $y = x^4$ is a cubic polynomial function

It was interesting to learn more about why the group did not manage to determine the equation of graph c, which might lead to better justification, despite touching on the issue. After the task completion, the interview revealed students' thoughts below when asked how confident they were with their answers ("I" for the interviewer).

I: How confident are you with your answer? Can you share your stories?

S1: I agree that a is of degree three, b is degree two. But c, I am confused.

T1: I get a suspicion that because the two functions are of degree two and three, c must be of degree four.

S1: Yes, but if c is x^4, it is not the same (with the shape of the graph). It is more similar to b, but it is steeper.

T1: Right, when I see S's graph of x^4, I believe there must be something more. There are more requirements. Not just a simple x^4.

S1: Yes, there must be more of it. But the thing is, we don't know what those are.

T1: Yes, the coefficients of a, b, c, d (from $ax^4 + bx^3 + cx^2 + dx + e$), we must know them, so the graph will be like c.

I: Do you think you need to know the equation of the function to solve this problem?

T1: Yes. I think we need it. Just to make sure it is really degree four.

R1: But again, we don't know how.

Students in Group 1 knew that they needed an equation to justify why graph c is of degree four. However, they could not come up with any clues about the equation. This failure to find the equation made them resort to the justification of what they could offer by giving an example of a four-degree polynomial function.

4.2. Group 2 (T2, T3, and S2)

The group started the solving process by reading the problem. The analysis stage was started by identifying that they needed to show if Andi's statement about the problem was accurate. It happened right after reading the sentence, "Andi stated that a was the graph of f, b was the graph of f', and c was the graph of f''. The sentence was read repeatedly during the discussion, showing the role of this sentence in refocusing students' process to get the answer. After identifying the objective, students' discussion led to their plan to find the answer. They planned to determine the equation of each function to know which function was the derivative of which.

They went to the implementation stage by determining the equation of graph b first. While trying to determine the equation, they got distracted by the shapes of each graph. T3 claimed that the function in graph b was the second derivative of f due to its simplest shape among all graphs. She described what she meant by simple as "having least ups and downs", which was agreed by her peers. It was shown that they used the perceptual demonstration to justify this claim. They further shifted their plan from finding the equation for each graph to observing the shapes. By observing the shapes of the rest of the two graphs, T2 claimed that c is the polynomial function of degree four and a is the polynomial function of degree three due to its shapes. This claim was agreed upon by T3 and S2, and this agreement led to their further claim that b is the derivative of a and a is the derivative of c. They made a claim based on the fact that the derivative of a cubic polynomial is a quadratic function. However, the empirical demonstration type of justification did not satisfy T3, as stated below.

T3: If this graph's (graph b) shape is like this, it is surely the last graph (the second derivative)

T2: Graph b (*emphasizing*).

T2: I think it is certain. This graph (graph a) looks the same with cubic polynomial function, so this is the second one (the first derivative).

T3: But certainly, we need to know the equation, right? So that we can explain it easily.

T2, S2: Yes.

They understood that using the shapes of the graphs to identify their relations might not be enough. T3 triggered the mathematical justification to a higher level by requesting them to return to the original plan, i.e., determining the equation of the functions. The excerpt below shows how they came up with the equation for graph c.

T2: I believe c is of degree four. That shape is degree four. But what is the equation?

T3: What about the stationary points? If this (pointing at the stationary points at x –axis) is 1 and -1 let's just write $(x - 1)$ and $(x + 1)$ first and see (writing $(x - 1)\ (x + 1)$).

S2: But that's degree two.

T3: Right. What to do? Should we just square them?

T2: I think we learnt before in polynomial function, right? (Trying to open her notes).

S2: Try squaring it, maybe?

T3: Like this? (Writing $(x-1)^2(x+1)^2$)

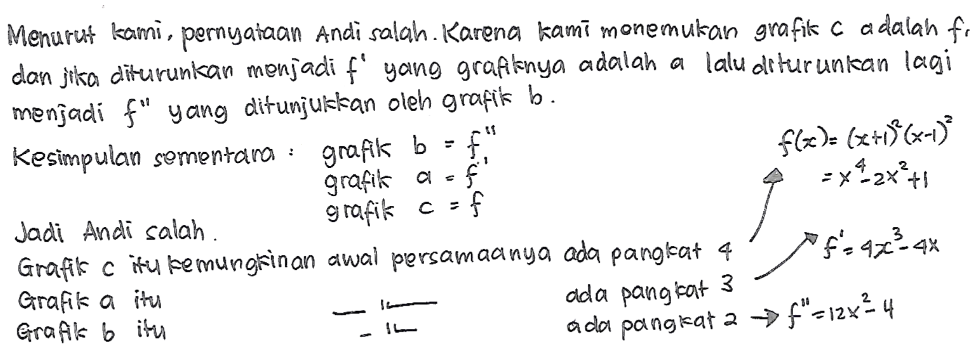
T2: How do we know it's true? So, it will be (multiplying the polynomials) $x^4 - 2x^2 + 1$.

T3: I think it is the same with the graph, guys. Plug in -1 or 1. We get 0.

T2: If $x = 0$? Wow, true! It is 1. I think we are correct, guys.

Group 2's dissatisfaction with merely using the graph shape led them to try establishing the equation of graph c. By having a degree four equation as the goal, they conjectured an equation by making use of the fact that the graph passed through $(-1{,}0)$ and $(1{,}0)$. The fact led to the equation having $(x-1)$ and $(x+1)$. Further, they knew they needed to justify the conjectured equation to the given graph by substituting several values of x. Their mathematical justification was a symbolic example with generalization as they used properties to find the equations without relying on examples or perceptual demonstration. Their discussions brought the equation of graph c and its first and second derivatives. They concluded by writing the final work, as shown in Figure 4.

Figure 4. Final work of Group 2


Menurut kami, pernyataan Andi salah. Karena kami menemukan grafik c adalah f, dan jika diturunkan menjadi f' yang grafiknya adalah a lalu diturunkan lagi menjadi f'' yang ditunjukkan oleh grafik b.

Kesimpulan sementara : grafik b = f''
grafik a = f'
grafik c = f

$f(x) = (x+1)^2(x-1)^2$
$= x^4 - 2x^2 + 1$

Jadi Andi salah.

Grafik c itu kemungkinan awal persamaanya ada pangkat 4
Grafik a itu — " — ada pangkat 3
Grafik b itu — " — ada pangkat 2

$f' = 4x^3 - 4x$
$f'' = 12x^2 - 4$

Translation: in our opinion, Andi's statement is wrong. Because we find that graph c is f, and its derivative is f' which is a, and its second derivative is f'' which is b.

Tentative conclusion: graph $b = f''$, graph $a = f'$, graph $c = f$.

So, Andi is wrong.

The equation of graph c is possibly of degree 4.

The equation of graph a is possibly of degree 3.

The equation of graph b is possibly of degree 2.

The CPS process in Group 2 showed all criteria of collaborative interaction. Either claims, triggers for justification, or justifications were made by each student. The group was synchronous in going through the CPS process, with claims made by students quite evenly. The negotiability was shown by no student dominating the claims or justification. Five justifications were done by Group 2, with two of them addressing the same claim (Table 3). No appeal to authority was observed, and students justified their claims using a symbolic example with generalization for some claims. The justifications around claim K3 showed how prior justification ("The derivative of a three-degree polynomial function is quadratic") was considered not enough and refined into a better justification (Determining the equation of each function).

Table 3. Claims and justifications made by Group 2

	Claims (by order of appearance)	Mathematical justification	Students' utterances/ activity
K1	b is the second derivative of f	Empirical or perceptual demonstration	"b has the least ups and downs"
K2	c is the polynomial function of degree 4, a is the polynomial function of degree 3	Empirical or perceptual demonstration	"If the shape is like this, it is four-degree polynomials"
K3a	b is the derivative of a and a is the derivative of c	Symbolic example without generalization	"The derivative of a three-degree polynomial function is quadratic"
K3b	b is the derivative of a and a is the derivative of c	Symbolic example with generalization	Determining the equation of each function
K4	$f(x) = x^4 - 2x^2 + 1$ is the function of graph c, $f'(x) = 4x^3 - 4x$, and $f''(x) = 12x^2 - 4$.	Symbolic example with generalization	Determining the derivatives using the derivative rule

Compared to Group 1, Group 2 managed to have better justification by coming up with graph c's equation to show it is a degree-four polynomial function. Students showed urgency in coming up with an equation through several responses below.

S2: It (determining the degree based on shapes) just doesn't feel right.

T2: Yes. I keep thinking about the equation. If not, maybe I cannot sleep tonight (laughing).

T3: Derivative is a challenging topic, right? I don't think the way we should answer is that simple.

T2: I think so. The graph scares me, actually. Like you don't know how to verify from it, so, I think I should go with the equation to make sure if our answer is true.

We see that the motivation for finding the equation was vivid, and students did come up with this plan at the beginning of the CPS process. Their process in finding the equation could be a good example of how the justification was not initially pursued formally (e.g., by using a general form of $ax^4 + bx^3 + cx^2 + dx + e$ and utilizing the given points) but evolved through conjecturing the equation and going backwards to verify the equation. The mathematical justification was refined through students' discussion and the urge to provide a more convincing answer.

The potential of the decision-making task in stimulating justifications in CPS was discussed from two perspectives: students' interaction and the variety of justifications. Students' interaction in solving the task collaboratively was obvious in both groups. Addressing this, students admitted that the task was difficult but doable as each student managed to contribute to the group and completed each other. Several studies discussed the difficulty level (Chiu, 2008; Chiu & Khoo, 2003; Taylor & McDonald, 2007) to promote interactions in CPS. The presence of teachers and researchers during CPS and the recorded discussion might promote the interactions. Looking at the strategies brought by both groups, the task did not allow for the division of effort to get the solution. Each group step was collaboratively completed and recorded in the single shared working space.

The task allowed for a variety of mathematical justifications. It allowed students to concentrate their discussion on agreeing or disagreeing with the explicit claim made by Andi in the task. The two groups showed that they needed to justify several claims to get the solution. Interestingly, the two groups supported the same claim with different mathematical justifications. For example, both groups decided that Andi's statement in the task was wrong. To support this, both groups came to a claim that "b is the derivative of a and a is the derivative of c". Group 1 agreed on using an example that the derivative of $y = x^4$ is a cubic polynomial function as their mathematical justification to support that claim, despite knowing that they need to come up with equations of the functions. On the other hand, Group 2 believed that this justification alone was insufficient and continued to determine the equation of each function. On that point, we considered that the difference in mathematics abilities between the two groups played a role. For example, the success of Group 2 in finding the equation of c was promoted by how T3 came up with the idea of utilizing the function's zeros. This idea might not appear if T3 did not have enough understanding of the relationship between zeros and the function. Further, the group played along well in conjecturing the equation and verifying the equation, not relying on T3.

The two groups' interaction was different in some aspects, although all criteria of collaborative interaction were apparent. In Group 1, we observed that triggers for justification mostly, although not all, came from S1 and R1, while T1 dominantly provided the justifications. On the other hand, each student in Group 2 somehow more dynamically exchanged roles as either trigger or justification providers. We considered the role exchange as one of the factors that affected how Group 2 managed to provide better justifications for their claims during CPS. The role exchanges allowed each student in Group 2 to think actively between questioning "Is this really true?" and providing justification for the answer, which might not be the case for Group 1. Additionally, this role-taking allowed a particular justification to be questioned again and then re-justified by the group, not by an individual member.

5. Conclusion

The decision-making task could help researchers see the mathematical justifications made by both groups. It allowed them to solve the problem collaboratively in a shared space. The task also promoted the discussion by concentrating on an explicit claim, encouraging students to justify further while trying to get the solution. The observed collaboration and justifications were also promoted by the adequate level of difficulty experienced by students, which might not be the case for others. Considering the task's difficulty level would be helpful for future studies utilizing the same task.

The task was given in a setting where students could not consult with the teacher or observer during the CPS process. In that case, any claims and justifications appeared naturally from the groups, allowing us to focus on their interactions and thinking without any support from others. However, we considered that the current task could be insufficient to see expected justifications if not observed since no specific types of justifications were requested in the written task. On that point, we recommended two efforts for a richer analysis. First, the task administration could be accompanied by by the teacher or researcher's scaffolding. The scaffolding might be developed based on the desired justification level and given whenever a situation of justification does not appear or when students' interaction goes in an undesirable direction. Another option is to add some guiding questions (as written scaffolding) for students transitioning from the lowest to the highest level of justifications.

The participants' selection in this study relied heavily on how they were considered low, average, or high in mathematics ability using students' report scores. The shift from using students' report scores to using a standardized test might result in different participants' selection and, further, on findings of the study. Although small in size, the study is worthwhile to be developed further. A similar task might be investigated using a larger sample to explore the task's potential better. Despite the limitations, it is hoped that the findings of this study offer helpful insights for researchers and teachers about a task stimulating mathematical justifications in CPS.

Acknowledgments

The authors would like to thank State University of Surabaya, Sampoerna University, the Center for Higher Education Funding (*Balai Pembiayaan Pendidikan Tinggi*), The Indonesia Endowment Funds for Education (*Lembaga Pengelola Dana Pendidikan*) for the support, and Dean Refangga Setiadi for his meaningful assistance.

References

Bieda, K. N., Conner, A. M., Kosko, K. W., & Staples, M. (Eds). (2022). *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof*. Springer International Publishing. https://doi.org/10.1007/978-3-030-80008-6

Brodie, K. (2010a). Pressing dilemmas: Meaning-making and justification in mathematics teaching. *Journal of Curriculum Studies*, 42(1), 27-50. https://doi.org/10.1080/00220270903149873

Brodie, K. (Ed.) (2010b). *Teaching Mathematical Reasoning in Secondary School Classrooms*. Springer US. https://doi.org/10.1007/978-0-387-09742-8

Carpenter, T. P., Levi, L., Berman, P. W., & Pligge, M. (2005). Developing algebraic reasoning in the elementary school. In T. A. Romberg, T. P. Carpenter & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 81-98). Routledge.

Chiu, M. M. (2008). Flowing toward correct contributions during group problem solving: A statistical discourse analysis. *Journal of the Learning Sciences*, *17*(3), 415-463. https://doi.org/10.1080/10508400802224830

Chiu, M. M., & Khoo, L. (2003). Rudeness and Status Effects During Group Problem Solving: Do They Bias Evaluations and Reduce the Likelihood of Correct Solutions? *Journal of Educational Psychology, 95*(3), 506-523. https://doi.org/10.1037/0022-0663.95.3.506

Chua, B. L. (2016). Examining Mathematics Teachers justification And Assessment of Students justifications. *Proceedings of the 40th Conference of the International*, *August*.

Chua, B. L. (2017). A Framework for Classifying Mathematical Justification Tasks. *CERME 10*, 115-122. https://hal.archives-ouvertes.fr/hal-01873071

Cirillo, M., Kosko, K. W., Newton, J., & Staples, M. (2016). *White Paper 2016: Conceptions and Consequences of What We Call Argumentation, Justification, and Proof. April.* https://doi.org/10.13140/RG.2.1.5132.0727

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, *23*(1), 99-122. https://doi.org/10.1007/BF00302315

de Villiers, M. (1990). The role and function of proof in Mathematics. *Pythagoras*, 24(August), 17-23.

Díez-Palomar, J., Chan, M. C. E., Clarke, D., & Padrós, M. (2021). How does dialogical talk promote student learning during small group work? An exploratory study. *Learning, Culture and Social Interaction*, *30*, 100540. https://doi.org/10.1016/j.lcsi.2021.100540

Dillenbourg, P. (1999). What do you mean by collaborative learning?. In P. Dillenbourg (Ed.), *Collaborative learning: Cognitive and computational approaches* (Vol. 1, pp. 1-19). https://telearn.archives-ouvertes.fr/hal-00190240

Ellis, A. B., Staples, M., & Bieda, K. N. (2022). *Justification Across the Grade Bands* (pp. 287-297). https://doi.org/10.1007/978-3-030-80008-6_23

Hamidy, A., & Suryaningtyas, S. (2016). Kemampuan Justifikasi Matematis Siswa SMP Pada Materi Segitiga. *Prosiding Seminar Nasional Pendidikan Matematika*, 978-989.

Heid, M. K., Hollebrands, K. F., & Iseri, L. W. (2002). Reasoning and Justification , with Examples from Technological Environments. *The Mathematics Teacher, 95*(3), 210-216.

Jaffe, A. (1997). Proof and the Evolution of Mathematics. *Synthese*, *111*(2), 133-146. http://www.jstor.org/stable/20117623

Jermann, P. (2004). Computer Support for Interaction Regulation in Collaborative Problem-Solving. *Proceedings of Computer Support for Collaborative Learning, CSCL 2002*, *340*, 601-602. https://telearn.hal.science/hal-00197373

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn Mathematics*. National Academy Press.

Küchemann, D., & Hoyles, C. (2006). Influences on students' mathematical reasoning and patterns in its development: Insights from a Longitudinal study with particular reference to geometry. *International Journal of Science and Mathematics Education*, 4(4), 581-608. https://doi.org/10.1007/s10763-006-9039-6

Lai, E. R. (2011). *Collaboration: A Literature Review*. Pearson's Research Reports.

McClain, K., & Cobb, P. (2001). An Analysis of Development of Sociomathematical Norms in One First-Grade Classroom. *Journal for Research in Mathematics Education*, *32*(3), 236-266. https://doi.org/10.2307/749827

Mueller, M. F. (2009). The co-construction of arguments by middle-school students. *Journal of Mathematical Behavior*, *28*(2-3), 138-149. https://doi.org/10.1016/j.jmathb.2009.06.003

NCTM. (2000). *Principles and Standards for School Mathematics*. Author.

Partanen, A. M., & Kaasila, R. (2015). Sociomathematical Norms Negotiated in the Discussions of Two Small Groups Investigating Calculus. *International Journal of Science and Mathematics Education*, *13*(4), 927-946. https://doi.org/10.1007/s10763-014-9521-5

Rott, B., Specht, B., & Knipping, C. (2021). A descriptive phase model of problem-solving processes. *ZDM - Mathematics Education*, *53*(4), 737-752. https://doi.org/10.1007/s11858-021-01244-3

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *15*(1), 3-31. https://doi.org/10.1016/S0732-3123(96)90036-X

Sowder, L., & Harel, G. (1998). Types of Students' Justifications. *The Mathematics Teacher*, *91*(8), 670-675. https://doi.org/10.5951/mt.91.8.0670

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, *31*(4), 447-462. https://doi.org/10.1016/j.jmathb.2012.07.001

Staples, M., & Conner, A. (2022). Introduction: Conceptualizing Argumentation, Justification, and Proof in Mathematics Education. In K. N. Bieda, A. Conner, K. W. Kosko, & M. Staples (Eds.), *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof* (pp. 1-10). Springer. https://doi.org/10.1007/978-3-030-80008-6_1

Stylianou, D. A., & Blanton, M. L. (2002). Sociocultural factors in undergraduate mathematics: the role of explanation and justification. *Proceedings of the Second Annual Conference on the Teaching of Mathematics*.

Tatsis, K. (2007). Investigating the influence of social and sociomathematical norms in collaborative problem solving. *Cerme 5*, *5*(November), 1321-1330.

Tatsis, K., & Koleza, E. (2008). Social and socio-mathematical norms in collaborative problem-solving. *European Journal of Teacher Education*, *31*(1), 89-100. https://doi.org/10.1080/02619760701845057

Taylor, J.., & McDonald, C. (2007). Writing in groups as a tool for non-routine problem solving in first year university mathematics. *International Journal of Mathematical Education in Science and Technology, 38*(5), 639–655. https://doi.org/10.1080/00207390701359396

Thanheiser, E., & Sugimoto, A. (2022). Justification in the Context of Elementary Grades: Justification to Develop and Provide Access to Mathematical Reasoning. In K. N. Bieda, A. Conner, K. W. Kosko, & M. Staples (Eds.), *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof* (pp. 35-48). Springer. https://doi.org/10.1007/978-3-030-80008-6_4

Vale, C., Widjaja, W., Herbert, S., Bragg, L. A., & Loong, E. Y. K. (2016). Mapping Variation in Children's Mathematical Reasoning: The Case of 'What Else Belongs?' *International Journal of Science and Mathematics Education*, *15*(5), 873-894. https://doi.org/10.1007/s10763-016-9725-y

Walton, D. (2001). Abductive, presumptive and plausible arguments. *Informal Logic*, *21*(2), 141-169. https://doi.org/10.22329/il.v21i2.2241

Whitacre, I., Azuz, B., Lamb, L. L. C., Bishop, J. P., Schappelle, B. P., & Philipp, R. A. (2017). Integer comparisons across the grades: Students' justifications and ways of reasoning. *Journal of Mathematical Behavior*, *45*, 47-62. https://doi.org/10.1016/j.jmathb.2016.11.001

Yackel, E. (2004). Theoretical perspectives for analyzing explanation, justification and argumentation in mathematics. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education, 8*(1), 1-18.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477. https://doi.org/10.2307/749877

∞

Namirah Fatmanissa

State University of Surabaya (Indonesia)
namirah.21027@mhs.unesa.ac.id | https://orcid.org/0000-0002-2734-1214

Tatag Yuli Eko Siswono

State University of Surabaya (Indonesia)
tatagsiswono@unesa.ac.id | https://orcid.org/0000-0002-7108-8279

Agung Lukito

State University of Surabaya (Indonesia)
agunglukito@unesa.ac.id | https://orcid.org/0000-0003-1277-5834

Ismail

State University of Surabaya (Indonesia)
ismail@unesa.ac.id | https://orcid.org/0000-0003-1969-5415

Recibido: 1 de enero de 2023
Aceptado: 8 de agosto de 2023

Utilización de una tarea de toma de decisiones: justificación matemática de los estudiantes en la resolución colaborativa de problemas

Namirah Fatmanissa @, Tatag Yuli Eko Siswono @,
Agung Lukito @, Ismail @

State University of Surabaya (Indonesia)

La importancia de la justificación matemática en la resolución colaborativa de problemas (CPS, por sus siglas en inglés) ha sido destacada por muchos estudios. Sin embargo, no todas las tareas pueden estimular la justificación matemática en el CPS. Este estudio explora el potencial de una tarea de toma de decisiones para facilitar la justificación matemática en el CPS en un tema de derivadas. La tarea fue diseñada de tal manera que había una afirmación explícita que debía ser validada, y el grupo debía decidir si la afirmación era verdadera. La tarea se administró a dos grupos de estudiantes de 12.º curso en Bandung, Indonesia. El Grupo 1 estaba compuesto por estudiantes con habilidades matemáticas alta, media y baja. El Grupo 2 estaba formado por estudiantes con habilidades alta y media. Durante el proceso de CPS, se permitió utilizar una calculadora no gráfica y se les animó a cuestionar cualquier opinión que consideraran dudosa. Se les informó que su proceso sería grabado y que su trabajo escrito sería recopilado al final de la sesión. Analizamos el potencial de la tarea desde dos perspectivas: las actividades de justificación matemática y las interacciones de los estudiantes dentro de cada grupo. Desde la perspectiva de la justificación matemática, identificamos las justificaciones hechas por los estudiantes en cinco niveles: sin justificación, apelación a la autoridad, demostración empírica o perceptual, ejemplo simbólico sin generalización y ejemplo simbólico con generalización. En cuanto a las interacciones de los estudiantes, las codificamos en tres criterios: interactividad, sincronicidad y negociabilidad. Además, codificamos qué estudiante iniciaba o proporcionaba las justificaciones matemáticas para entender mejor las interacciones. Los resultados mostraron que la tarea permitió diversas justificaciones matemáticas. Se observó que ambos grupos se centraron en validar la afirmación explícita de la tarea y resolvieron la tarea con éxito. Durante el proceso de CPS, el Grupo 1 logró ofrecer ejemplos simbólicos sin generalización como su justificación para la afirmación final. Por otro lado, el Grupo 2 logró acordar una mejor justificación mediante ejemplos simbólicos con generalización para su afirmación final. Desde la perspectiva de las interacciones, las justificaciones del Grupo 1 fueron ofrecidas principalmente por los estudiantes con habilidades altas, mientras que las justificaciones del Grupo 2 fueron ofrecidas de manera alternada entre los estudiantes. Como factor adicional al potencial de la tarea, consideramos que la dificultad adecuada percibida por los estudiantes fue un factor de apoyo para las discusiones. Sugerimos examinar la efectividad de la tarea en una muestra más grande.