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The Role Played by Extra-mathematical Connections in the Modelling Process

El papel de las conexiones extra-matemáticas en el proceso de modelización

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Abstract ∞ Literature review in Mathematics Education on connections makes evident the necessity of delving into extra-mathematical connections. In this line, we answer the question: What types of mathematical connections are required to develop the modelling process? To this end, we consider three theoretical references: Onto-Semiotic Approach, Extended Theory of Connections, and Mathematical Modelling Cycle from a Cognitive Perspective, following a methodology used in two theoretical articulations previously developed by the authors between these frameworks, based on the use of models for the analysis of mathematical activity proposed by the Onto-Semiotic Approach for connections and the modelling process. The results of this analysis consist of, on one hand, the evidence of the types of intra- and extra-mathematical connections intervening in the different phases of the modelling cycle and, on the other hand, the proposal of a more detailed classification of mathematical connections.

Keywords ∞ Extra-mathematical connection; Mathematical modelling; Onto-semiotic approach

Resumen ∞ La revisión de la literatura en Educación Matemática sobre las conexiones evidencia que es necesario profundizar sobre las conexiones extra-matemáticas. En esta línea, se responde a la pregunta ¿qué tipos de conexiones matemáticas son necesarias para desarrollar el proceso de modelización? Para ello, se consideran tres referentes teóricos: el Enfoque Onto-Semiótico, la Teoría Ampliada de las Conexiones, y el Ciclo de Modelización Matemática desde una Perspectiva Cognitiva; siguiendo una metodología usada en dos articulaciones teóricas desarrolladas previamente por los autores entre estos marcos, basada en el uso del modelo de análisis de la actividad matemática propuesto por el Enfoque Onto-Semiótico para las conexiones y el proceso de modelización. Los resultados de este análisis consisten en, por una parte, la evidencia de los tipos de conexiones intra- y extra-matemáticas que intervienen en las diferentes fases del ciclo de modelización y, por otra parte, la propuesta de una clasificación más detallada de conexiones matemáticas.

Palabras clave ∞ Conexión extra-matemática; Enfoque Onto-Semiótico; Modelización matemática

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1. Introduction

Various investigations in Mathematics Education consider the study of mathematical connections to be important. The focus of these investigations has been mainly on intra-mathematical connections; however, little has been delved into the role played by extra-mathematical connections.

Another trend focused on relating students' mathematical knowledge and competencies with the solving of real-world problems is mathematical modelling (Kaiser, 2020). Usually, from research in connections, that of 'Modelling' type is considered as the paradigmatic example of extra-mathematical connection.

In this line, we consider it relevant to pose the following question: What types of mathematical connections are required to develop the modelling process? To answer it, we take into consideration the theoretical articulations proposed in literature between the Onto-Semiotic Approach (OSA), the Extended Theory of Connections (ETC; see Rodríguez-Nieto et al., 2023), and the mathematical modelling process (MOD; see Ledezma et al., 2023). In these articulations, we made evident that a conglomerate — formed by *mathematical practices*, *primary mathematical objects* (activated in such *practices*), *semiotic functions* (linking these *primary objects*), and other *mathematical processes* — is necessary for mathematical connections to be activated and for the modelling process to be developed. Therefore, in this study, we take as a base the importance of analysing this conglomerate underlying both processes to delve deeper into the relationship between mathematical connections (particularly, extra-mathematical ones) and modelling.

In the theoretical articulations mentioned above, a combination of two references is handled (ETC-OSA and MOD-OSA), thus, because of the complexity of the mathematical activity underlying the modelling process, in this article, we use the three theoretical references aiming a more detailed analysis of modelling. Finally, we refine some of the theoretical references used, based on the results emerging from the analyses performed.

2. Theoretical Framework

Subsections 2.1, 2.2, and 2.3 synthesise the three theoretical references considered in our investigation, and subsection 2.4 synthesises the articulation developed among them.

2.1. Onto-Semiotic Approach

The OSA considers that, to describe mathematical activity, it is essential to take into consideration the *objects* involved in such activity and the semiotic relationships between them (Font et al., 2013). Mathematical activity is modelled in terms of *practices*, a *configuration of primary objects*, and *processes* that are activated by the *practices*. In this theory, a *mathematical practice* is considered as a sequence of actions regulated by institutionally established norms and guided towards an objective (usually, solving a problem). In the OSA ontology, the term *object* is used in a broad sense to refer to any entity that is, in some way, involved in a *mathematical* *practice* and that can be isolated from the rest of the mathematical activity and identified as a unit. In the OSA terminology, a distinction is made between *primary* and *secondary objects* (Font et al., 2013); however, when in this article we use the term *object*, we are referring to *primary mathematical objects*. For example, when carrying out and evaluating a problem-solving *practice*, we can identify the use of different languages (verbal, graphical, symbolic, etc.), which are the ostensible part of a series of definitions, propositions, and procedures that are involved in the argumentation of the solution of a problem. Problems, languages, definitions, propositions, procedures, and arguments are considered as the six *primary mathematical objects* which jointly form the *configuration of primary objects*. The term *configuration* is used to designate a heterogeneous set or system of *objects* that are interrelated. Every *configuration of primary objects* can be viewed from both a personal and institutional perspective, which leads to the distinction between *cognitive* (personal) and *epistemic* (institutional) *configurations of primary objects*.

Mathematical objects that intervene in *mathematical practices* and those that emerge from them can be considered from the perspective of the following ways of being present in mathematical activity, which are grouped into facets or dual dimensions (Font & Contreras, 2008; Font et al., 2013):

- Extensive/Intensive: Intensive *objects* correspond to those collections or sets of entities, of any nature, which are produced either extensively (by enumerating the elements when possible) or intensively (by formulating the rule or property that characterises the membership of a class or type of *objects*).

- Expression/Content: *Objects* can be participating as representations or as represented *objects*.

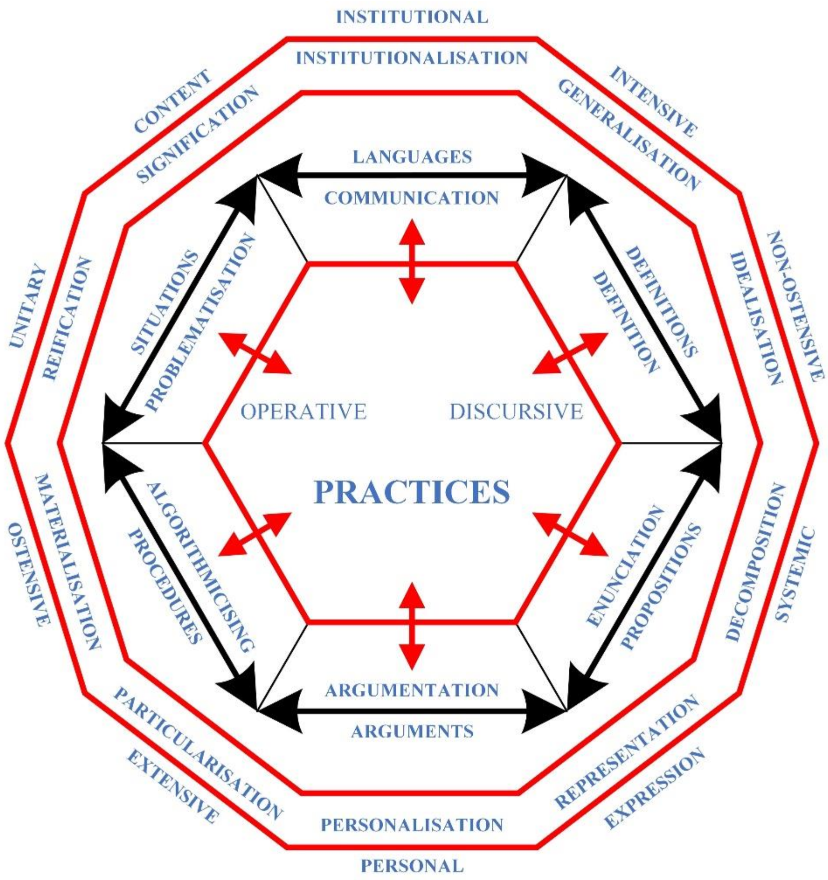
- Personal/Institutional: Institutional *objects* emerge from systems of shared *practices* within an institution, while personal *objects* emerge from specific *practices* of an individual.

- Ostensive/Non-ostensive: Something that can be shown directly to another person, versus something that cannot be directly shown on its own and must, therefore, be complemented by another something that can be directly shown.

- Unitary/Systemic: *Objects* can participate in *mathematical practices* as unitary *objects* or as a system.

The use and/or emergence of the *primary objects* of the *configuration* (problems, languages, definitions, propositions, procedures, and arguments) takes place through the respective *mathematical processes* of communication, problematisation, definition, enunciation, elaboration of procedures (algorithmicising, routinisation, etc.), and argumentation (by applying the process-product duality). Meanwhile, the dualities described above give rise to the following *processes* (Font et al., 2013): institutionalisation-personalisation; generalisation-particularisation; analysis/decomposition-synthesis/reification; materialisation/concretion-idealisation/abstraction; expression/representation-signification (see Figure 1).

Figure 1. Onto-semiotic representation of mathematical knowledge



Source: Adapted from Font & Contreras (2008, p. 35).

This list of *processes* is derived from the typology of *primary objects* and dual facets used as tools to analyse mathematical activity in the OSA. Although some of the *processes* considered as important in mathematical activity are contemplated, it is not intended to include all the *processes* involved in such activity. This is because, among other reasons, some of the most important *processes*, such as problem-solving and mathematical modelling, are *macroprocesses* that involve more elementary *processes*, such as representation, argumentation, idealisation, etc.

The notion of *semiotic function* allows *practices* to be related to the *primary objects* that are activated in them. A *semiotic function* is a triadic relationship between an antecedent (initial expression/object) and a consequent (final content/object) established by a subject (person or institution) according to certain criterion or correspondence code.

The theoretical tools described above allow us to analyse mathematical activity through different types of didactic analyses (see Breda et al., 2021), which we describe in greater detail in section 3.

2.2. Extended Theory of Connections

The ETC considers the following 10 types of mathematical connections that have emerged from the results of different research on this topic:

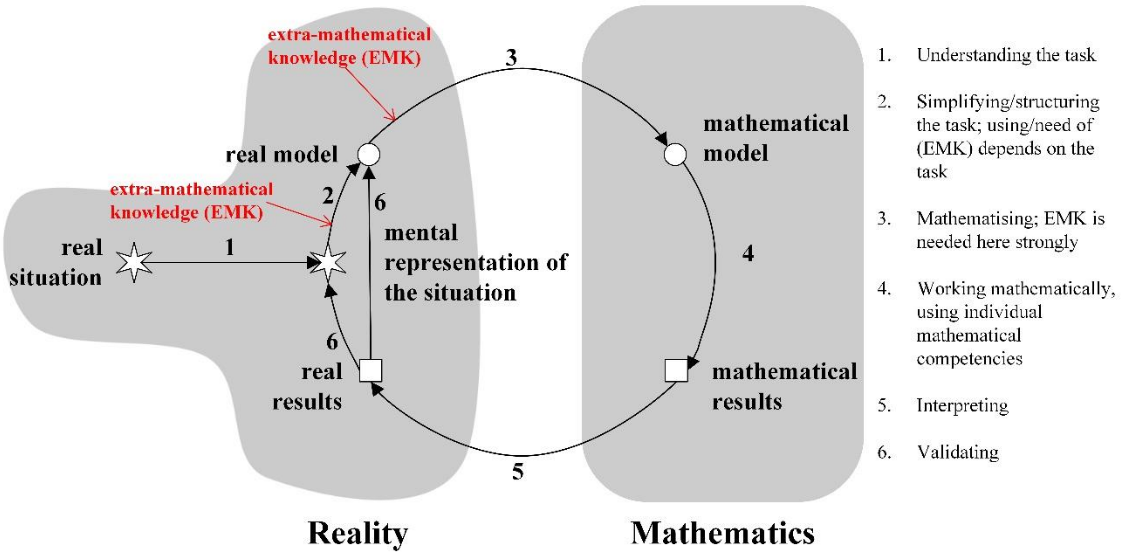
- *Instruction-oriented*: A concept *C* is connected with previous concepts (Businskas, 2008).
- *Modelling*: This connection corresponds to the relationships between mathematics and real life and is made evident when an individual solves non-mathematical or application problems where he/she must propose a mathematical model or expression (Evitts, 2004).
- *Procedural*: This connection is identified when an individual uses rules, algorithms, or formulas to solve a mathematical concept. This connection is of the form *A* is a procedure to work a concept *B* (García-García & Dolores-Flores, 2021a).
- *Different representations*: This connection is identified when an individual represents mathematical objects using equivalent (from the same register) or alternative (from different registers) representations (Businskas, 2008).
- *Part-whole*: This connection occurs when logical relationships are established in two ways. The first refers to the generalisation relation of the form *A* is a generalisation of *B*, and *B* is a particular case of *A*; the second is that the inclusion relation is given when a mathematical concept is contained in another (Businskas, 2008).
- *Feature*: This connection is identified when an individual expresses some characteristics of concepts or describes their properties in terms of other concepts that make them different or similar to others (Eli et al., 2011).
- *Reversibility*: This connection occurs when an individual starts from a concept *A* to obtain a concept *B* and reverses the process, starting from concept *B* to return to concept *A* (García-García & Dolores-Flores, 2021a).
- *Meaning*: This connection is identified when an individual attributes meaning to a mathematical concept or uses it in solving a problem (García-García & Dolores-Flores, 2021b).
- *Implication*: This connection is identified when a concept *A* leads to another concept *B* through a logical relationship (Businskas, 2008).
- *Metaphorical*: This connection is understood as the projection of properties, characteristics, etc., of a known domain to structure another, less known domain (Rodríguez-Nieto et al., 2022).

We must stress that these connections can be intra-mathematical, identified when 'they are established between ideas, concepts, procedures, theorems, representations, and mathematical meanings among themselves' (Dolores-Flores & García-García, 2017, p. 160, authors' translation); and extra-mathematical, identified when 'a mathematical concept or model is related to an application or in-context problem or vice versa. They include the connections between mathematical contents with other curricular disciplines and with daily-life situations' (Dolores-Flores & García-García, 2017, p. 161, authors' translation). For example, in the case of a connection of 'Metaphorical' type, if it is a *grounding metaphor*, it is extra-mathematical, and if it is a *linking metaphor*, it is intra-mathematical.

2.3. Mathematical Modelling Process

Mathematical modelling is a process that describes the translation of a problem from a real context into mathematics and its subsequent return to reality. In the literature on modelling, different cycles have been designed to explain this process (Borromeo Ferri, 2006), and different perspectives have emerged on its implementation in the classroom (Preciado et al., 2023). In the work of Ledezma et al. (2023), the Mathematical Modelling Cycle from a Cognitive Perspective (MMCCP), proposed by Borromeo Ferri (2018), is adopted, which allows explaining the phases and transitions that an individual goes through to solve a modelling problem (see Figure 2).

Figure 2. Mathematical modelling cycle from a cognitive perspective



Source: Adapted from Borromeo Ferri (2018, p. 15)

In Borromeo Ferri (2018), the cycle in Figure 2 is described as follows: the *real situation* is a problem taken from reality and represented in any form (through texts and/or pictures) to the solver; the *mental representation of the situation* is generated from the understanding and mental reconstruction of the task, establishing associations of the individual with the *real situation*; the *real model* is the simplification and structuration of the mental image, which requires external representations (diagrams, figures, etc.); the individual's *extra-mathematical knowledge* adds some considerations to the context of the real situation according to personal experience and additional information that can be obtained; the *mathematical model* takes into consideration the mathematical objects that allow the *real situation* to be explained (Abassian et al., 2020), and is the result of the mathematisation of the *real model* and the contributions of the individual's *extra-mathematical knowledge*; by working with the *mathematical model*, *mathematical results* will be obtained that must be interpreted in the context of the *real situation* to obtain *real results*.

2.4. ETC-OSA and MOD-OSA Articulations

In this study, we considered as references two theoretical articulations previously conducted by the authors based on the OSA: one using the Extended Theory of Connections (ETC-OSA; e.g., Rodríguez-Nieto et al., 2023) and another using the mathematical modelling process (MOD-OSA; see Ledezma et al., 2023). Both studies followed the *Networking of Theories* methodology.

In both theoretical articulations, we integrated a type of analysis designed for any mathematical activity (proposed by the OSA) with analysis tools for a specific type of mathematical activity (connections and modelling). In metaphorical terms, we can say that, as a result of both articulations, both the connection and the modelling process can be understood as an iceberg where, to explain how an individual performs a modelling process and establishes connections, we need to take into consideration the submerged part of the iceberg, which consists of a conglomerate of *mathematical practices*, *primary mathematical objects* activated in them, *semiotic functions* that relate these *primary objects*, and other *mathematical processes*. In both articulations, we made evident the necessity of this conglomerate both for mathematical connections to be activated and for the modelling process to be developed.

2.5. Research Objectives

We pose the following objectives to answer the research question posed:

1. To jointly use the theoretical references OSA, ETC, and MOD to explain the complexity of the mathematical activity underlying the modelling process.

(2) To refine the constructs of the theoretical references used as a consequence of the results of the analyses performed.

3. Methodology

In methodological terms, this is a reflective-on-theory study, based on the analysis of the expert solving of a modelling problem. To do this, we initially developed a mathematical narrative about this expert solving, to which we applied the analysis model of mathematical activity proposed by the OSA, which consists of the following types of didactic analysis: 1) Identification of the *mathematical practices* in this narrative; 2) Identification of the *primary mathematical objects* activated in these *practices*; 3) Establishment of the plot of *semiotic functions* between the *mathematical objects*; and 4) Identification of the *mathematical processes* involved in these *practices*. After these four analyses, 5) we related the *practices*, *primary mathematical objects*, *semiotic functions*, and *processes* with the intra- and extra-mathematical connections, using the categories of connections proposed by the ETC; and, finally, 6) we related all these analyses to the phases of the MMCCP. In summary, we tried to jointly conduct analyses similar to those reported in Rodríguez-Nieto et al. (2023) for connections and in Ledezma et al. (2023) for modelling.

4. Analysis of the Expert Solving of a Modelling Problem

For this study, we considered the *Bales of Straw Problem* (see Figure 3) as a context for reflection, whose choice is justified because it is considered a paradigmatic example to explain the MMCCP (see Borromeo Ferri, 2018, pp. 15-17).

Figure 3. Bales of Straw problem

Bales of straw

At the end of the summer one can see a lot of bales of straw. Bales of straw on the picture are piled up in this way that in the bottom line are five, in the next four, then three, then two and on the top one ball.

Try to find out, how high this mountain of bales of straw is.

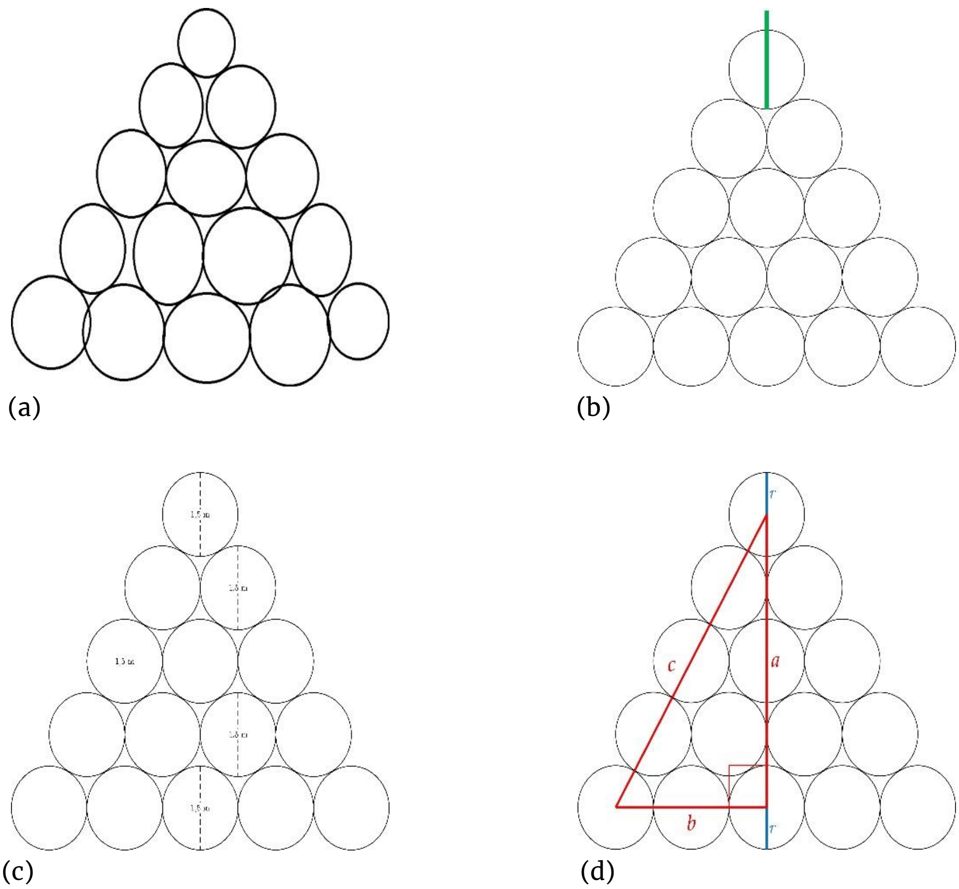
Source: Adapted from Borromeo Ferri (2018, p. 14).

In the description of Borromeo Ferri (2018, p. 16), two *mathematical models* are proposed to solve the problem in Figure 3: estimation of height and Pythagorean theorem. Since the first *mathematical model* was already used by the authors to develop the MOD-OSA articulation (Ledezma et al., 2023), we used the Pythagorean theorem to solve the problem in this article. Hence, this solving protocol developed by the authors was considered as the expert and, therefore, institutional solving. In the following subsections, we detail how we applied the mathematical activity analysis model proposed by the OSA (described in section 3) for this solving.

4.1. Mathematical Narrative

Firstly, we developed a mathematical narrative about the expert solving of the problem. The solver began by reading the wording of the problem, which requires calculating the height of a mountain of bales of straw. To do this, he/she created a sketch based on the description of the situation, representing the mountain as a diagram showing the five rows of bales of straw (see Figure 4a). Then, using graphing software, he/she made a more idealised representation of the situation, in which he/she converted the bales into cylinders and only considered a basal face; next, he/she estimated the height of the woman in the picture, attributing an approximate value of 1.7 metres due to the fact that she is German, and visually compared this height to the diameter of one bale of straw (see Figure 4b). In this way, the solver concluded that the height of the bale may be a little less than that of the woman, attributing to it an approximate diameter of 1.5 metres and adding this value to the second representation of the situation (see Figure 4c) on which he/she continued working. With these initial estimates, he/she determined that a plausibly realistic result would be a total height of slightly less than five diameters (< 7.5 metres). Next, the solver drew a right triangle on the representation of the situation, whose vertices were: in the centre of the ball of straw at the top, in the centre of the central bale at the base, and in the centre of the bale from one of the ends of the base (see Figure 4d). In this way, he/she formed a right triangle in the representation of the situation and considered that the height of the mountain of bales of straw could be solved by applying the Pythagorean theorem.

Figure 4. (a) Initial sketch, (b) comparison between the height of the woman and of one bale of straw, (c) real model, and (d) mathematical model.



To obtain the unknown height of the right triangle (a), he/she determined that, if the shortest leg (b) is 2 diameters long ($2d$) and the hypotenuse (c) is 4 diameters long, by applying the Pythagorean theorem, he/she obtained:

$$a^2 = c^2 - b^2$$

$$a^2 = (4d)^2 - (2d)^2$$

$$a^2 = 16d^2 - 4d^2$$

$$a^2 = 12d^2$$

$$a = \sqrt{12d^2}$$

$$a = 2\sqrt{3}d$$

Thus, the solver determined that, to calculate the total height of the mountain of bales of straw, he/she had to add the two remaining radii (or one diameter) of the upper and lower bales of the mountain to the value of the largest leg (a), thus obtaining a height $h = 2\sqrt{3}d + d$. Since the solver had estimated the diameter (d) of each bale of straw to be 1.5 metres, he/she substituted this numerical value into the mathematical result for the height of the mountain, so that, if $d = 1.5$, then:

$$h = 2\sqrt{3}(1.5) + 1.5$$

$$h = 3\sqrt{3} + 1.5$$

Thus, the solver obtained a total height of approximately 6.7 metres for the mountain of bales of straw. Finally, he/she compared the result obtained (6.7 metres) with the estimation initially proposed for the total height of the mountain (< 7.5 metres), concluding that the result was plausible and that there was no error in the calculations. We presented this expert solving to three mathematicians, who validated it as a correct solution to the problem.

4.2. Mathematical Practices

Mathematical practices happen over time, therefore, a good way to infer them is by narrating what happened (Font et al., 2013). In fact, the narration of *mathematical practices* is the discourse that a teacher uses when he/she wants to explain to others what he/she has done in his/her lessons. The narrative of the previous subsection introduces a temporal order of mathematical actions that is almost a list of *mathematical practices*. Therefore, secondly, we identified the *mathematical practices* that can be performed to solve the *Bales of Straw Problem*, based on the mathematical narrative above, which are presented in the first column of Table 1.

4.3. Epistemic Configuration of Primary Objects

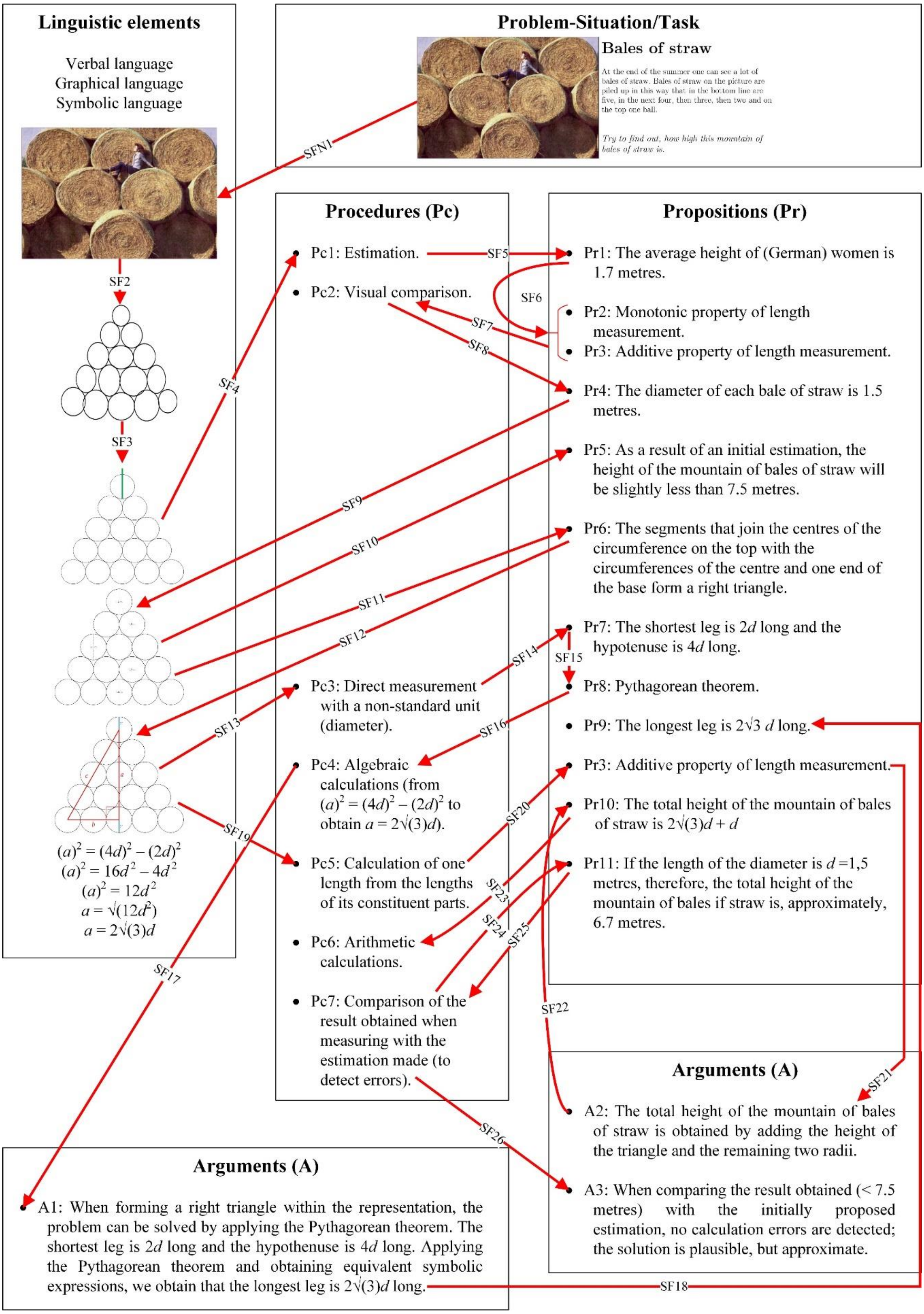
When the narrative is made, a series of temporally ordered actions appear from which *mathematical practices* are inferred. Now, in this narrative, the 'main characters of the mathematical story' (problems, representations, propositions, etc.) also appear, which are the *primary mathematical objects*. To obtain the *configuration of primary objects*, we need to complete the list with the 'secondary or supporting characters' who are also present, but who may not appear in the narrative. Therefore, thirdly, we identified the *primary mathematical objects* activated in the *mathematical practices*, which are presented in the second column of Table 1.

4.4. Plot of Semiotic Functions

Primary mathematical objects activated in *mathematical practices* do not intervene in isolation but are interrelated. Fourthly, we established the plot of *semiotic functions* between the *primary mathematical objects* involved in the expert solving of the problem, which is presented in the third column of Table 1 and in greater detail in

Figure 5. This plot is derived from the narrative and the first and second columns of Table 1, although we only selected those that we considered most relevant.

Figure 5. Plot of semiotic functions



4.5. Mathematical Processes

We can talk about a *process* as a sequence of *practices*, cognitive/metacognitive processes, instructional processes, social processes, etc. Also, we can say that a *practice*, being a sequence of actions, is a *process*, etc. In the OSA, since there is a certain overlap between the notions of *practice* and *process*, it has been decided to differentiate between *practice* (sequence of actions performed subject to mathematical rules), procedure (*primary object* that is a rule that indicates the steps to be followed), and *process* (which involves the time factor and that each member of the sequence takes part in determining the next one). Given this overlap, in the narrative made and in the list of *practices* that has been derived, some *processes* are already found, which can be complemented by applying the process-product duality to the *primary objects* and, finally, adding other *processes* that do not easily appear in the discourse of *practices* in the first phase (for example, *processes* of generalisation, instantiation, signification, etc.). This description of *processes* begins by establishing the *semiotic functions* (uncategorised connection *processes*) that relate the *primary objects* of the *configurations,* and is completed with the *processes* in the fourth column of Table 1. To do this, it is important to understand, in addition to the establishment of a *semiotic function* can be considered as a connection *process*, the fact that *semiotic functions* are situated in the expression/content duality since, depending on the antecedent and the consequent, the establishment of a *semiotic function* can also be interpreted as another *process*. For example, if we ask a student what the term 'derivative' means to him/her, the *semiotic function* that is established between 'derivative' and his/her 'response' can be understood as a signification *process*, which later allows us to consider this connection as one of 'Meaning' type.

4.6. Mathematical Connections

The *semiotic functions* described in the third column of Table 1 are connection *processes* between *primary mathematical objects* that are not categorised, but the other *process* that intervenes allows it to be typified (enclosed in a category or type of connection). The next step is to categorise them according to the typology of connections proposed by the ETC. Furthermore, we classified these connections as intra- (when the two ends of the connection are considered as *mathematical objects*) or extra-mathematical (when one of the two *objects* is considered extra-mathematical). These connections are presented in the fifth column of Table 1.

4.7. Relation with the MMCCP

Finally, we related the conglomerate of *practices*, *primary objects*, *semiotic functions*, *processes*, and connections with the phases and transitions of the MMCCP, as shown in the last column of Table 1. We stress that, in some phases of the MMCCP, we used the term 'state' to individualise them, which is because they correspond to an input/output of a portion of mathematical activity within the modelling process (Ledezma et al., 2023).

Table 1. Epistemic configuration of primary objects

Practices	Objects	Semiotic Functions	Processes	Connections*	Phases of the MMCCP
	– Problem.				Real situation (state).
P1: *Reads the task.*	– Problem: *Determine the height of the mountain of bales of straw.*	SFN1**	– Signification/ Understanding/ Communication. – Problematisation.	– Meaning (EM). – Different representations (the picture is related to the text; EM).	Real situation → Mental representation of the situation.
P2: *Makes simplified representations on the situation.*	– Representation (R) R1: *Figure 4a.*	SF2	– Representation (graphical). – Simplification/ Idealisation: *Consider the bales first, ignoring their other attributes. Convert the bales of straw into cylinders, consider only one basal face (circle), and then the circumference…*	– Different representations (EM). – Part-whole (eliminative abstraction; EM).	Real situation → Mental representation of the situation → Real model.
	– R2: *Figure 4b.*	SF3	*… Consider only the shape of the woman and the bales, ignoring their other attributes. Convert the woman into a line segment.* – Representation (graphical).	– Idealising (dematerialises; EM). – Different representations (EM).	
P3: *Makes estimations on the situation (height of the woman, diameter of the bales of straw, total height of the mountain).*	– Procedure (Pc) Pc1: *Estimation.* – Definition (D) D1: *Arithmetic mean (implicit definition).*	SF4	– Algorithmicising (Pc1 is almost automatic).	– Procedural (IM).	Real situation → Mental representation of the situation → Real model.
	– Proposition (Pr) Pr1: *The average height of (German) women is 1.7 metres.*	SF5	– Enunciation.	– Feature (EM).	
	– Pr2: *Monotonic property of length measurement (implicit proposition).* – Pr3: *Additive property of length measurement (implicit proposition).*	SF6		– Implication (IM).	
	– Pc2: *Visual comparison.* – D2: *Cylinder, base, circumference, diameter (implicit definitions).*	SF7	– Algorithmicising (Pc2 is almost automatic).	– Procedural (EM).	
	– Pr4: *The diameter of each bale of straw is 1.5 metres, approximately.*	SF8	– Enunciation.	– Feature (EM).	
	– R3: *Figure 4c.*	SF9	– Representation (graphical and symbolic).	– Different representations (IM).	
	– Pr5: *As a result of an initial estimation, the height of the mountain of bales of straw will be slightly less than 7.5 metres.*	SF10	– Enunciation.	– Feature (EM).	

Practices	Objects	Semiotic Functions	Processes	Connections*	Phases of the MMCCP
	– R3.				Real model (state).
P4: *Joins the centres of three circumferences forming a right triangle on the graphical representation of the situation.*	– D3: *Right triangle.* – Pr6: *The segments that join the centres of the circumference on the top with the circumferences of the centre and one end of the base form a right triangle.*	SF11	– Instantiation. – Enunciation.	– Part-whole (IM). – Feature (IM).	Real model → Mathematical model.
	– R4: *Figure 4d.*	SF12	– Representation (graphical and symbolic).	– Different representations (IM).	
P5: *Calculates the lengths of the right triangle applying the Pythagorean theorem.*	– Pc3: *Direct measurement with a non-standard unit (diameter).*	SF13	– Algorithmicising (Pc3 is almost automatic).	– Procedural (IM).	Mathematical model → Mathematical results.
	– Pr7: *The shortest leg is 2d long and the hypotenuse is 4d long.*	SF14	– Representation (symbolic). – Generalisation. – Enunciation.	– Different representations (IM). – Feature (IM).	
	– Pr8: *Pythagorean theorem.*	SF15	– Instantiation. – Enunciation.	– Part-whole (IM). – Feature (IM).	
	– Pc4: *Algebraic calculations (from* $(a)^2 = (4d)^2 - (2d)^2$ *to obtain* $a = 2\sqrt{3}d$*).*	SF16	– Algorithmicising (Pc4 is almost automatic).	– Procedural (IM). – Different representations (IM).	
	– Argument (A) A1: *When forming a right triangle within the representation, the problem can be solved by applying the Pythagorean theorem. The shortest leg is 2d long and the hypothenuse is 4d long. Applying the Pythagorean theorem and obtaining equivalent symbolic expressions, we obtain that the longest leg is* $2\sqrt{3}d$ *long.*	SF17	– Argumentation.	– Implication (IM).	
	– Pr9: *The longest leg is* $2\sqrt{3}d$ *long.*	SF18	– Enunciation.	– Feature (IM).	
P6: *Makes calculations to obtain the total height of the mountain of bales of straw.*	– Pc5: *Calculation of one length from the lengths of its constituent parts.*	SF19	– Algorithmicising (Pc5 is almost automatic).	– Procedural (IM).	Mathematical model → Mathematical results.
	– Pr3: *Additive property of length measurement (implicit proposition).*	SF20		– Part-whole (IM).	
	– A2: *The total height of the mountain of bales of straw is obtained by adding the height of the triangle and the remaining two radii.*	SF21	– Argumentation.	– Implication (IM).	
	– Pr10: *The total height of the mountain of bales of straw is* $2\sqrt{3}d + d$.	SF22	– Representation (symbolic). – Enunciation.	– Feature (IM).	

Practices	Objects	Semiotic Functions	Processes	Connections*	Phases of the MMCCP
	– Pr10.				Mathematical results (state).
P7: Assigns real values to the total height of the mountain of bales of straw and assesses whether the solution makes sense in the context of the problem.	– Pr6: *Arithmetic calculations.*	SF23	– Algorithmicising (Pc6 is almost automatic). – Instantiation (*d*=1.5 m).	– Procedural (IM). – Part-whole (IM).	Mathematical results → Real results.
	– Pr11: *If the length of the diameter is d=1.5 metres, therefore, the total height of the mountain of bales of straw is 6.7 metres, approximately.*	SF24	– Enunciation.	– Feature (IM).	
	– Pc7: *Comparison of the result obtained when measuring with the estimation made (to detect errors).*	SF25	– Algorithmicising (Pc7 is almost automatic).	– Procedural (IM).	
	– A3: *When comparing the result obtained (< 7.5 metres) with the initially proposed estimation, no calculation errors are detected; the solution is plausible, but approximate.*	SF26	– Argumentation (validation of the solution, taking into consideration the context).	– Implication (EM).	
	– Pr10.				Real results (state).

* IM = intra-mathematical connection; EM = extra-mathematical connection.
** Plot of semiotic functions that allow understanding the problem and give meaning to each of the mathematical terms — not detailed.

5. Discussion of Results

Table 1 is the result of applying the types of didactic analyses described in section 4. In this section, since the interest of our study is in the relationship between extra-mathematical connections and the modelling process, we will emphasise on discussing the analysis of the uncategorised connections (third column of Table 1), in their typification (fifth column), and in the modelling cycle (sixth column).

5.1. On the Semiotic Functions

A first aspect to highlight is that, although in the third column of Table 1 we emphasised on the *semiotic functions* (SF) that relate the *primary objects*, these are not restricted only to this type of *mathematical objects*. For this reason, we began by considering a plot of SFs (SFN1) that allows the solver to understand the problem, relate it to an external physical situation that can be mathematised, and also focus on the term ‘height’ (verbal language) and on the picture of the wording of the task (graphical language) as starting points for its solving. However, we did not detail this plot, given its complexity.

A second aspect to highlight is that, although SF2 is presented in Figure 5 as a SF between representations, it could also be understood as a SF that relates the problem to a schematic representation. Finally, in some cases, the *primary object* intervenes implicitly and with little awareness of the solver, so perhaps it would not be necessary to consider a certain SF. An example of this is SF20 where, although the solver applies the additive property of lengths measurement (Pr3), he/she is surely not aware that he/she is doing it.

5.2. On the Mathematical Connections

Mathematical connections can be applied as a set of connections (for example, when considering that the entire last column of Table 1 is an extra-mathematical connection of 'Modelling' type) or as a single connection (for example, when considering that SF3 falls under the category of connection between different representations). In fact, the concrete connection is described by a SF in the third column and is typified in the fifth column according to the 10 types of connections proposed by the ETC. As it is a typology of connections given *a priori*, some problems appear when it is applied to a particular connection, that is, to a SF. Just like in Rodríguez-Nieto et al. (2023), we chose, from the outset, to apply the previous typology of the ETC and, if this was not possible, to generate a new type of connection. Below, we discuss some of the problems we had when applying the 10 types of connections proposed by the ETC, which we resolved by expanding the field of application of some of these connections, and we only chose to generate a new type of connection.

A first problematic connection is that of 'Different representations' type, which seems to have been defined with the different representations of the mathematical object function as a paradigmatic example. SF3 fits easily with this typology (different equivalent representations of the same object), although there is already some problem because it is not clear that the object represented (Figure 4a) is a *mathematical object*, unless it is considered that the ostensive/non-ostensive duality of the OSA also acts, which converts this representation into that of an ideal *mathematical object*. Therefore, in this analysis, when we applied this type of connection, we have dispensed with the requirement that the object represented be mathematical, since we have considered that this duality also operates. In other words, in the case of SF3, it is not enough to consider the expression/content duality of the OSA and limit ourselves to considering that there is a connection between representations, but it is necessary to consider that the ostensive/non-ostensive duality that enables, in both cases, but especially in Figure 4b, that the circles/circumferences are considered dematerialised ideal circles/circumferences, which leads to considering a new type of connection, as we detail in section 6.

Another problem with this type of connection is that it relates two different representations, which raises an important philosophical problem about the relationship between the mathematical object and its representation. With no intention to delve into this philosophical problem, we have a physical, external, and structured situation, which we can describe mathematically or see as a concretisation of mathematical ideas, and this situation, thanks to a verbal and graphical language, allows the problem to be posed. But let us suppose that in the proposed problem there is no picture and that we only have a verbal description of the *mathematical object* problem, in this case, part of the wording of the task could be considered as a verbal representation of the situation, which could be connected with a geometric representation (in this case, connections between alternate representations). In other words, when the SF relates a *primary object* that is not a representation (a problem, a procedure, a definition, a proposition, or an argument) with a representation, since the *primary object* is made ostensive through a certain language, we consider that this SF falls under the connection of 'Different representations' type, as in the case of SF12. This type of connection is a good example of how, even a connection considered punctual, is a conglomerate of *mathematical practices*, *primary objects*, *semiotic functions*, and *processes*. Indeed, the connection of 'Different representations' type implies, at a minimum, a *process* of representation of the second representation, and a *process* of signification of the first representation. Being a type, it is made concrete in a particular SF which related *primary objects* activated in some *practice*.

A second problematic connection is that of 'Part-whole' type, described by the ETC in two ways: first, as the particular-general relationship and, second, as that the inclusion relation is given when a mathematical concept is contained in another. In this case, we have SFs that we considered as connections of 'Part-whole' type, in which one of these two interpretations was not clearly given. For example, in the procedure Pc5: Calculation of one length from the lengths of its constituent parts, the additive property of lengths measurement is implicitly used (SF20), in this case, there is a relationship between the parts and the whole that cannot be considered the particular-general relationship not is it clear that it is when one mathematical concept is contained in another. Therefore, in accordance with the extensive/intensive duality of the OSA, we applied the connection of 'Part-whole' type to the SFs that relate an intensive to an extensive or vice versa.

A third problematic connection is that of 'Feature' type, which seems inspired by the classical conception of concept formation. According to this conception, a concept is made up of a series of necessary and sufficient attributes, in such a way that all examples of the concept have common attributes and no counterexample has them all. In other words, concepts would have the structure of logical classes of the form $C = R(x, y, \dots)$, where C would be the concept, (x, y) its attributes, and R the relationship between such attributes. The connection of 'Feature' type seems intended for one established between a concept and one of its necessary attributes. Now, in the analysis of the connections involved in solving the problem, we have expanded this perspective by categorising this type of connection, in many cases, those in which the final end is a proposition.

Finally, we highlight that we did not find connections of the 'Instruction oriented' (since it is assumed that the solver of the problem has all the prior knowledge necessary for its solving), 'Reversibility', nor 'Metaphorical' types.

5.3. On the MMCCP

As mentioned at the beginning of section 4, the choice of the *Bales of Straw Problem* is justified because it is considered a paradigmatic example to explain the MMCCP. In other words, it would be a problem that adjusts to the MMCCP to make it operational, so in our study there were no difficulties in identifying and/or determining the phases and transitions that an individual goes through for its solving. However, the representation of the cycle (see Figure 2), with set characteristics (similar to a Venn diagram), suggests a strict separation between 'real world' and 'mathematics' that we did not appreciate as such in the expert solving of the problem. In the mathematical narrative described in subsection 4.1, the R3 representation (Figure 4c) is considered, in terms of the MMCCP, as the *real model* of the situation that, in turn, would be part of the 'real world'; however, R3 could also be considered within 'mathematics', since it includes idealised dematerialised circumferences and segments.

6. Conclusions

The first conclusion of our study is related to the paradigmatic extra-mathematical connection proposed by the ETC, namely, that of 'Modelling' type. In this sense, we propose to rethink it in two types of connections: 'Modelling' and 'Application of models'. This new proposal for types of connections is justified by the difference established by Blum (2002) between both processes, which, although they represent relationships between the intra- and extra-mathematical, have different foci. On one hand, modelling focuses from the extra- to the intra-mathematical, emphasising the mathematisation of the extra-mathematical context, which is where this process is validated. On the other hand, applications focus from the intra- to the extra-mathematical, emphasising the mathematical object involved, and are validated based on intra-mathematical considerations. A *mathematical model* intervenes in both processes which, from a didactic point of view, may or may not be preestablished in a mathematical teaching and learning process.

The second conclusion of our study is that an extra-mathematical connection can be established in a unitary or systemic way. When we say that the connection of 'Modelling' type is extra-mathematical, we are establishing it as a macro-connection (unitary), in which many elements intervene to establish it (systemic). In turn, if some of the transitions of the modelling cycle are seen from a unitary perspective, they can also be considered as extra-mathematical connections.

The third conclusion of our study is that a systemic look at the connections makes it evident that they are based on a conglomerate of *mathematical practices*, *primary objects*, *semiotic functions*, and *processes*. Now, Table 1 shows that, even when we look at the connections in a unitary way, we also must consider this conglomerate. For its part, a more unitary look at the connections (third and fifth columns) allows us to see that their typology can also be intra- or extra-mathematical, depending on whether the origin or the end is considered extra-mathematical. However, unlike what happens in systemic connections, in unitary connections, this characterisation is, in some cases, problematic. For example, the distinction of the antecedent (or the consequent) as extra- or intra-mathematical is very clear in the case of connections of 'Metaphorical' type, depending on whether it is a *grounding* or *linking metaphor* type. Now, there are cases in which it is considered to be an extra-mathematical connection, where the starting (or final) point is not so clear that it is extra-mathematical. For example, SF3 is considered extra-mathematical because the starting point is a schematic representation of an extra-mathematical situation.

Finally, regarding the 10 types of connections proposed in the ETC, the results of our study allowed us to develop two proposals. On one hand, we propose to extend the connection of 'Feature' type to include propositions, by making the following modification to the name and definition of this type of connection:

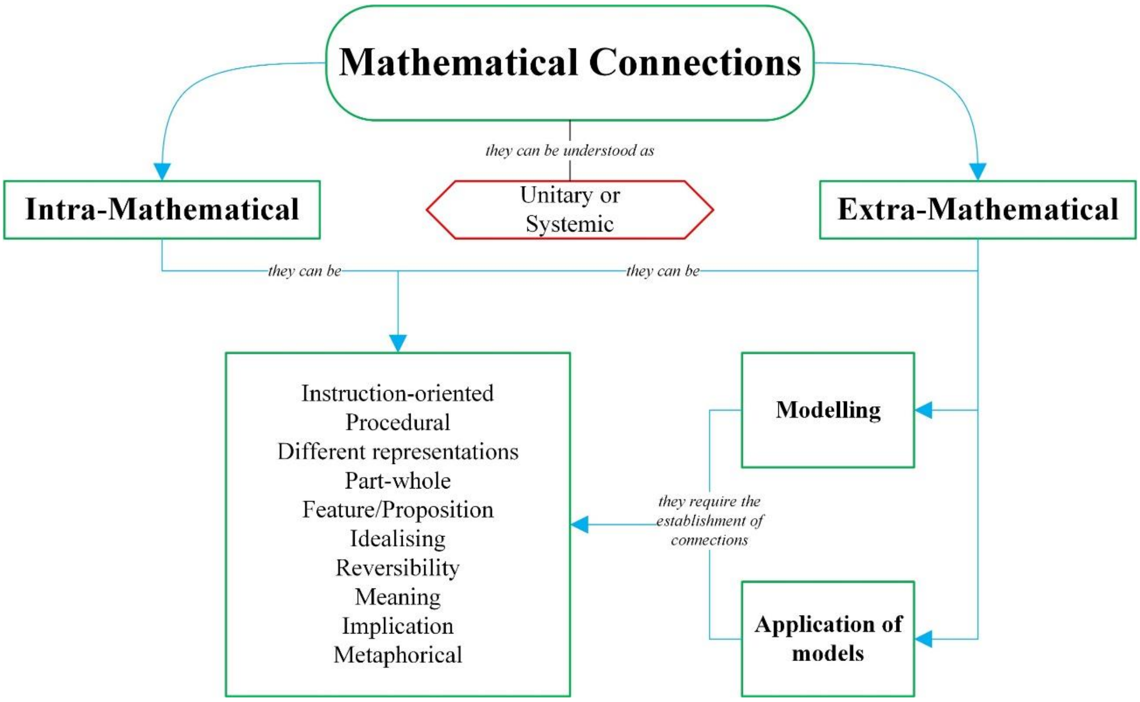
- *Feature/Proposition*: This connection is identified when an individual expresses some characteristics of concepts or described their properties in terms of other concepts that make them different or similar to others (Eli et al., 2011), *or states/uses a proposition in which such concept has a determining role*.

On the other hand, we propose a new type of connection, namely, the 'Idealising' type. In the rows of P2: Makes simplified representations of the situation, in Table 1, we can see that an eliminative abstraction intervenes, considered under the category of the connection of 'Part-whole' type, and idealisation. From the OSA perspective, this last connection is not located in the extensive/intensive duality (which would allow it to be included in the connection of 'Part-whole' type), but it is located in the ostensive/non-ostensive duality, which justifies a new type of connection that we describe as follows:

- *Idealising*: This connection relates an ostensive to a non-ostensive. Its function is to dematerialise the ostensive and turn it into an ideal mathematical object (for example, the rounded drawings in Figure 4a are considered circles/circumferences).

Taking into consideration these conclusions and the previous section, Figure 6 presents our proposal of a typology for mathematical connections.

Figure 6. Proposal of a typology for mathematical connections



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El papel de las conexiones extra-matemáticas en el proceso de modelización

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La investigación en Educación Matemática considera importante el estudio de las conexiones matemáticas dado su papel en los procesos de enseñanza y aprendizaje matemáticos. Sin embargo, la revisión de la literatura sobre las conexiones evidencia que es necesario profundizar sobre las conexiones extra-matemáticas, pues la mayoría de las investigaciones se enfocan en el estudio de las conexiones intra-matemáticas. Otra tendencia en Educación Matemática es la inclusión de la modelización, ya que permite relacionar los conocimientos y competencias de los estudiantes con la resolución de problemas del mundo real. Además de ser un proceso relevante de la actividad matemática, la modelización es considerada, en la literatura sobre las conexiones, como el ejemplo paradigmático de conexión extra-matemática. En esta línea, en este artículo se busca responder a la pregunta ¿qué tipos de conexiones matemáticas son necesarias para desarrollar el proceso de modelización? Para ello, se consideran tres referentes teóricos: el Enfoque Onto-Semiótico (EOS), la Teoría Ampliada de las Conexiones (TAC), y el Ciclo de Modelización Matemática desde una Perspectiva Cognitiva (CMMPC). En este estudio se sigue una metodología similar a la utilizada en dos articulaciones teóricas previamente desarrolladas por los autores basadas en el uso del modelo de análisis de la actividad matemática propuesto por el EOS. En este contexto, primero, se tomó un problema de modelización paradigmático y se elaboró una narrativa matemática (considerada como el protocolo de resolución experta); segundo, se identificaron las *prácticas matemáticas* en esta narrativa; tercero, se identificaron los *objetos matemáticos primarios* emergentes en estas *prácticas*; cuarto, se estableció la trama de *funciones semióticas* entre estos *objetos*; quinto, se identificaron los *procesos matemáticos* intervinientes en las *prácticas*; sexto, se relacionaron las *prácticas matemáticas*, los *objetos matemáticos primarios*, las *funciones semióticas*, y los *procesos matemáticos* con las conexiones intra- y extra-matemáticas que propone la TAC y, finalmente, todos estos análisis se relacionaron con las fases del CMMPC. Los resultados de este análisis permitieron evidenciar los tipos de conexiones intra- y extra-matemáticas que intervienen en las diferentes fases del ciclo de modelización cuando se resuelve un problema de este tipo, así como la necesidad de considerar nuevos tipos de conexiones no contemplados por la TAC. Finalmente, se propone una clasificación más detallada de conexiones, la cual consiste, por una parte, en refinar la conexión extra-matemática de tipo «Modelado» en dos tipos: «Modelización» y «Aplicaciones»; y, por otra parte, en la nueva conexión de tipo «Idealizadora».