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Mathematical reasoning in linear systems learning: a higher education exploratory teaching experiment with prospective teachers

Razonamiento matemático en el aprendizaje de sistemas lineales: una experiencia de enseñanza exploratoria en educación superior con futuros profesores

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Abstract ∞ Higher education students show difficulties on concepts in linear systems, due to procedural-dominated teaching practices. This emphasizes the need to develop students' mathematical reasoning using an exploratory teaching approach to promote learning with understanding. This qualitative and interpretative study analyzes the mathematical reasoning that prospective teachers, attending the degree in mathematics, use in solving research tasks involving linear systems, proposed throughout an exploratory teaching experiment and how this context contribute to their learning. Data collection includes participant observation of the teaching experiment classes, and written work of the proposed tasks. The results show that prospective teachers have evolved positively in their understanding and capacity of mathematical reasoning, and in the linear systems learning. The evidenced advantages of this experiment may contribute to a reflection on this integration to improve educational contexts, including preservice teacher education, to overcome their difficulties in learning and develop their knowledge for teaching.

Keywords ∞ Mathematical reasoning; Exploratory teaching; Linear systems equations; Undergraduation in mathematics; Pre-service teacher education

Resumen ∞ Estudiantes de educación superior muestran dificultades en conceptos de sistemas lineales, debido a prácticas de enseñanza dominadas por procedimientos. Este hecho enfatiza la necesidad de desarrollar su razonamiento matemático utilizando una enseñanza exploratoria para promover aprendizaje con comprensión. Este estudio cualitativo e interpretativo, analiza el razonamiento matemático que futuros profesores, cursando la licenciatura en matemática, utilizan en la resolución de tareas de investigación que involucran sistemas lineales, propuestos en una experiencia de enseñanza exploratoria, y cómo este contexto contribuye a su aprendizaje. Los datos incluyen observación participante de clases de la experiencia y trabajo escrito de las tareas propuestas. Los resultados muestran que los futuros profesores evolucionaron positivamente en su comprensión y capacidad de razonamiento matemático, y en aprendizaje de sistemas lineales. Las ventajas evidenciadas por la experiencia contribuyen a una reflexión sobre esta integración para mejorar los contextos educativos, incluida la formación inicial de profesores, para superar sus dificultades en el aprendizaje y desarrollar sus conocimientos para la enseñanza.

Palabras clave ∞ Razonamiento matemático; Enseñanza exploratoria; Sistema de ecuaciones lineales; Licenciatura en Matemática; Formación inicial de profesores

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1. INTRODUCTION

The theme of linear systems is embracing and fundamental in mathematics, as it begins in basic education and is applied to mathematical problems in middle and higher education, and also enables interdisciplinary approaches in different educational and scientific contexts (Souza & Simmer, 2014). In mathematics degree, its approach is common both in linear algebra, with a theoretical and conceptual focus, and in numerical calculus, involving computational properties and characteristics. The importance of linear systems in higher education and the frequent difficulties that students show, on complex concepts in this content and to relate equations to their solution (Possani et al., 2010), due to procedural-dominated teaching practices, emphasize the need to promote their mathematical reasoning development, recognized as an essential skill for their success in learning with understanding (Domingos, 2003; Stylianides & Stylianides, 2007). To this development, recent curriculum guidelines, at the various levels of mathematics teaching, highlight the skills of representing, communicating and argumentation (Ministério da Educação - Secretaria da Educação Básica, 2018; National Council of Teachers of Mathematics [NCTM], 2000).

As student learning depends on the pedagogical practices adopted, in particular the type of tasks and the way to propose them (Boston & Smith, 2011; Breen & O'Shea 2019), it is required to envision an alternative teaching and learning approach, contrary to the procedures memorization and transmission of concepts, with a focus on mathematical reasoning development. In this sense, research (Ponte et al., 2012; 2020) has highlighted the importance of exploratory teaching (Menezes et al., 2012), in which students, in interaction with their colleagues and teachers, carry out research activities that significantly contribute to the understanding and consolidation of mathematical concepts, and mathematical reasoning development, including argumentation.

However, the pedagogical practices in the context of exploratory teaching that promote students' mathematical reasoning, especially in the approach on mathematical contents of linear systems, are not yet known or experienced by teachers. This limitation is especially problematic in higher education, whose investigation is still scarce, especially in Brazil and in the context of pre-service teacher education, conditioning their future practices. In this context, it was considered relevant to carry out a teaching experiment with prospective mathematics teachers that were attending to the numerical calculus course of Mathematics degree, in order to promote their mathematical reasoning in the study of linear systems, through solving exploratory tasks (Ponte, 2005) in an exploratory teaching context.

Thus, this study may contribute to understand the evidenced potential of this exploratory teaching experiment, integrating mathematical reasoning, to improve educational contexts, including preservice teacher education. Helping mathematics prospective teachers (PTs) to overcome their difficulties in learning of linear systems could influence the way they will teach this mathematical topic (Albuquerque et al., 2006). The study aims to analyze (i) the mathematical reasoning that mathematics prospective teachers attending the degree in mathematics use in solving research tasks involving linear systems, proposed throughout an exploratory teaching experiment, and (ii) how this context contributed to their learning.

2. Theoretical Background

2.1. Mathematical reasoning

Mathematical reasoning (MR) is assumed to make justified inferences, from previous information, and can be typified as: inductive, abductive or deductive (Jeannotte & Kieran, 2017; Ponte et al., 2020).

Inductive reasoning (IR) and Abductive reasoning (AR) involve conjecturing, the IR by making assumptions, envisioning the discovery of a rule or law of formation and its generalization, through observation, testing of particular examples and identification of regularity patterns and AR part of an unprecedented or unusual event and seeks an explanation for its occurrence. Deductive reasoning (DR), recognized as logical, involves the validation or invalidity of the argumentation of inferences produced. This method involves propositional calculus by performing operations or relationships between propositions (Alencar Filho, 1975).

Basically, the logical connectives, the conjunction, the disjunction and the conditional are considered during the chain of assertions, as well as the relations of implication (=>) and equivalence (<=>), established by rules of inference, to elaborating an argument. In this ideology, syllogisms characterize consistent arguments, highlighting the modes of affirmation, negation, the hypothetical syllogism (transitivity) and the disjunctive.

Table 1. Reasoning processes (Ponte et al., 2020, p. 10)

Conjecture	*Can have as base* - observation; - construction; - transformation of previous knowledge; - combinations of observation, construction and transformation.	*Can take forms like* - Identify a possible solution for a problem; - Formulate a strategy to solve a problem.
Generalize		*Can take forms like* - recognize a pattern or a common property to a set of objects; - extend the domain of validity of a property to a wider set of objects.
Justify	*Can have as base* - definitions; - axioms, properties, general principles; - representations; - combinations of definitions, properties and representations.	*Can take forms like* - logical coherence; - use of generic examples; - use of counterexamples; - by exhaustion; - absurd

According to Ponte et al. (2020), the concept of MR also involves reasoning processes, including conjecture, generalize, and justify. Justification is considered essential in DR, although the generalization and conjectures formulation of a general nature is also a process inherent to this type of reasoning. In Table 1, some elements in which these MR processes can be based on and forms they can take are described.

2.2. Exploratory practices in learning of linear systems

Previous research has pointed out students' main learning difficulties of linear systems, particularly in formulating and solving linear equations systems and relate them to their graphical representation (Possani et al., 2010). Ponte et al. (2009) also categorized them as: i) understanding the notion of its meaning and the nature of its solution; ii) mastery of the resolution and correct execution processes until obtaining the solution; and iii) ability to solve problems represented in verbal language, based on the translation into algebraic language and on the interpretation of the solution according to the given conditions.

Thus, in teacher education, the didactic contexts in courses of specific content constitute a curriculum component (Pereira & Mohr, 2017). The mathematical tasks that university students engage in, and the way they are proposed in class, influence how they stimulate MR and the development of learning concepts with understanding (Domingos, 2003; Henriques, 2012; Ponte, 2005). This literature highlights that carrying out research tasks is desirable, giving them opportunity to formulate initial questions, use multiple representations and different solving strategies, favor reflection and discussion of meanings between teachers and students to share knowledge or doubts, and in MR formulate conjectures and generalizations, by testing, and validate it by presenting justifications.

This requires that teachers adopt an exploratory teaching approach as a teaching and learning strategy (Menezes et al., 2012), centered on students' work with dynamic and constructive approaches, when they are involved in carrying out mathematical tasks of an exploratory nature. According to the authors, this approach is structured in phases: (i) task *introduction* by the teacher, ensuring the students correct interpretation and their familiarization with the requested context, promoting their involvement in solving it; (ii) students' *autonomous work*, individually or in small groups, monitored by the teacher to ensure productive participation in the exploration process, in order to achieve knowledge discovery and construction by establishing connections with prior knowledge, and supporting them in any doubts through questioning; (iii) *collective discussion*, aiming that students share, compare and argue their ideas and resolutions, and a *final synthesis* of the main mathematical ideas involved in the task.

In the mathematics degree, the study of linear systems curricular content is directed to the courses of linear algebra, with a theoretical and conceptual focus, and numerical calculus involving computational properties and characteristics that are also common in the other disciplines of the course. Given its scope, it is up to the mathematics teacher to prepare an effective pedagogical work based on the theoretical, conceptual and operational aspects inherent to this content.

Current research focused on cognitive processes has considered essential to study different representations to give meaning to mathematical concepts (Cuesta et al., 2016). According to Henriques and Ponte (2014, p. 276) "mathematical representations are strongly related to mathematical reasoning due to its important role in teaching and learning mathematics and, consequently, in the development and understanding of reasoning processes". Thus, a diversity of representations (verbal, algebraic, graphic), which constitute potential elements for investigations and reflections in the classroom is relevant to consider in the teaching and learning of linear systems. Furthermore, the structure of a linear system involves principles of mathematical logic such as the logical connective 'and', represented by the key {, characterizing the conjunction between the algebraically described propositions. That is, a set of equations that must be solved concomitantly, whose analysis of the solution includes classifications involving relationships of implication or equivalence, which can generate conclusions based on tautologies or contradictions.

According to (Boldrini, 2000), knowledge is essential to the study of linear systems:

1) knowledge about the meaning and nature of its solution. A linear system can be defined as a set of equations with linear characteristics, which do not have transcendental operators or products between variables, whose concomitant resolution of equations results in a conclusion about its solution, being classified as *possible* or *impossible* when it has or does not have a solution. If possible, can have a single solution, called *determined*, or infinite solutions, called *undetermined*.

2) knowledge in solving it. There is the possibility of using direct methods, involving a finite number of operations, or indirect or iterative methods contemplating the generation of a sequence of approximate solutions in which the accuracy of the results depends on the performed iterations. Thus, the equivalence concept between systems is intrinsic. Some parameters are also usual for the classification of the solution, such as the 'Determinant', 'Rank' and 'Nullity'. The Determinant is a real number obtained from the square matrix elements, and when associated with the matrix of coefficients of a system, its non-zero result is characteristic of the 'Possible and Determined' case. The Rank of a matrix, scaled by rows, corresponds to the number of non-nulls and indicates the number of their rows or linearly independent columns. If the Rank of the extended or complete matrix of the system (involving coefficients of variables, independent or constants terms) is different from that of the coefficients, the system is classified as impossible. Nullity is obtained by the difference between the number of columns of the system coefficients matrix (number of variables) and its Rank, and indicates the number of free variables, whose value is not determined by the system, characterizing its degree of freedom.

3) knowledge on the interpretation of the solution. The understanding of algebraic language involves the correlation between a verbal situation and its representation through a mathematical model. The solution verification is inherent to the conclusion, involving the analysis of the conditioning of a possible and determined system, considering that poorly conditioned cases can generate numerical inconsistencies, that is, totally discrepant solutions from small numerical perturbations caused in their coefficients.

3. METHODOLOGY

3.1. Study context

This qualitative and interpretive study (Bogdan & Biklen, 1994) is focused on the analysis of the PT's work in solving research tasks proposed on an exploratory teaching experiment, carried out in a numerical calculus course of the 3rd year of Mathematics degree, at the preservice teacher education in Brazil. The teaching experiment aims at promoting mathematical reasoning in the learning of concepts and strategies for solving linear systems. This experiment took place over two weeks, in four classes (2 hours each) powered by the second author, having already approached the numerical solving through the use of direct and iterative methods and the analysis on the conditioning of a linear system. The PTs participants of the study (6 males and 4 females), who are attending to this course, had already taken the linear algebra course including concepts, definitions and properties of linear systems.

The study focuses on the following research questions that were set to answer the aim of the study:

1) what types and processes of mathematical reasoning do PTs use in solving exploratory tasks involving linear systems? And what difficulties do they show in its use?

2) what are the learning carried out by the PTs and how the teaching experiment contributes to their understanding of the linear systems concepts and procedures?

The described following proposed research tasks were carried out in sequence, created by the authors, intentionally to engage PTs' in MR processes that are the focus of the study, and to develop the specific knowledge of linear systems:

Task 1 (T1) admits as a general principle the variety of solving strategies. It includes questions aimed at formulating generalization based on observation, exploring the meaning and nature of the solution, based on solving three equivalent linear systems for a final description of this concept. To understood that equivalent systems have the same solution and that can be obtained using elementary operations or linear combinations between their lines.

Task 2 (T2) establishes a classification based on identification of characteristics. It aims an investigation considering the solving processes, in which the determinant is used as a parameter for the analysis of the linear system solution, looking ahead the recognition of cases involving determination, indeterminacy and no solution.

Task 3 (T3) includes questions that request the justified identification of the truth or falsity of a mathematical statement, and contemplates the interpretation of the solution, based on its verification in a set of linear equations. As an answer, the recognition of a false situation and the proposition of a true one, duly justified, are expected.

Task 4 (T4) considers questions that request or encourage the answers justification, solving strategies, or mathematical statements, which enable their diverse nature, namely based on logical coherence. Focusing on the solution interpretation, establishing a correlation between verbal and algebraic language, and verifying the solution, it is expected an analysis in which the linear system resulting from the proposed situation is characterized as poorly conditioned.

Task 5 (T5) includes questions that encourage establishing an organization of objects based on the identification of their characteristics. It intends to achieve solving processes, direct methods, equivalence between systems, and the identification of rank and nullity, for the identification of classification as Determined, Possible and Indeterminate or Impossible;

Task 6 (T6) includes the possibility of using several solving strategies involving a variety of representations, based on the exploration of solving processes, direct and indirect methods, equivalence between systems, and different representations, expecting interpretation as a result of the solution in the context of numerical solving of linear systems.

The classes dynamics followed the described phases of exploratory teaching, in which the PTs were organized into three groups (G1, G2, G3) in the autonomous work of tasks, including moments of discussion between them and the teacher. In the final collective discussions, conducted by the teacher and with the participation of the three groups, they shared and discussed their solutions, the concepts were taken up in accordance with the needs identified by the PTs. That is, attending to a demand generated by the identified errors, the doubts expressed, and the inconclusive reasoning presented. Here, the teacher prepared a final synthesis of the contents covered, involving the essential knowledge of linear systems (Boldrini, 2000).

3.2. Data collection and analysis

Data collection included participant observation (Bogdan & Biklen, 1994) of classes, with video recording of students' discussions, and PTs' written work on the proposed task. The descriptive and interpretative data analysis (Cohen et al., 2007) is focused on PTs' learning of linear systems concepts and procedures, on the mathematical reasoning processes they used to solve the proposed research tasks, and how the exploratory teaching context contributed to their learning. It is based on the theoretical references of mathematical reasoning types (AR, DR, IR) and processes (conjecture, generalize, justify), and the essential knowledge of linear systems (meaning and nature of its solution, namely: Possible and Determined System–PDS; Possible and Undetermined System- PIS; Impossible System IS).

In the next section, we present the results of the analysis organized by the six tasks performed by the PTs. We illustrated the results with excerpts of PTs written work and oral discussions of each group, whose names are not mentioned to ensure confidentiality.

4. RESULTS: TASKS EXPLORATION

Task1. This first task, in Figure 1, focused on the presentation of three linear systems. This is the beginning of the investigation since undergraduate students are asked to solve and represent the solutions, and express their understanding of the equivalence between linear systems, being able to use different representations. It is necessary to present the characteristics based on their previous knowledge.

Figure 1. Task 1 statement

Consider the linear systems below:

a) $\begin{cases} 2x-2y=2 \\ x+y=3 \end{cases}$ b) $\begin{cases} 3x-3y=3 \\ 5x+5y=15 \end{cases}$ c) $\begin{cases} x-y=1 \\ y=1 \end{cases}$

Solve them and represent the solutions. Then answer the following questions by presenting justifications: What characteristic(s) do you identify as common among systems? What can you conclude about systems that have this (those) common characteristic(s)? Will they be valid for any set of linear systems? Based on this and your knowledge in this regard, is it possible to establish a rule? Justify.

Initially, PTs directed their efforts to the algebraic solving of each system, without realizing that it was a case of equivalence, namely systems that had the same solution. Therefore, they solved each case individually. Although the solving method or the representation form were not specified, the three groups (G1, G2 and G3) solved in a similar way, by performing elementary operations in order to isolate a variable and, by retro substitution, obtaining the value of the other variable. Only G2 referred to the graphic solution, but did not represent it. Regarding the common characteristics between the systems, G3 answered with reference to the concept of equivalent equations.

> G2: Are systems that have a solution and determined (PDS), as graphically they are straight lines concurrent, which means that they have only a solution.

> G3: We can take as a rule that the equations, when they are proportional or equivalent, will have the same result.

The three groups used a solving strategy and G2 also performed a conjecture about PDS. The graphic identification of possible solutions, as well as the formulation of strategies, correspond to the act of conjecturing that is associated with AR. The elaboration of a conjecture of a general nature is based on AR and IR, as they solved the system to confirm the emerged hypothesis and extend it to other cases. However, in this case students considered only proportionality, in which an equation being multiplied in both members of equality by a real constant different from zero does not have its solution changed. It is verified the occurrence of an invalid justification, as they sought to relate the systems only considering the equivalence between equations. In turn, the equivalence between linear systems is broader, and involves the linear combination between equations.

In this sense, when asked by the professor if the proportionality between the equations was the only common characteristic between the systems and about the establishment of a rule, G2 mentioned the performance of elementary operations between the lines of the system, naming it as 'scaling', intending to obtain a triangular system whose resolution is simplified.

> G2: To identify these possible solutions and establish a rule, we can resort to scaling that does not change the system solution. To classify the system that is scaled, just look at two elements: the last line of the fully scaled system and the number of unknowns compared to the number of equations given in the system.

At the end of the task, the students concluded about the equivalence of the systems, without a mathematical formalization and without presenting a definition of this relationship. The reasoning processes evidenced in the first activity were: conjecture, through observation and solving; and justify, although this last process occurred in an incomplete way, not compatible with the DR, only with reference to general principles and representations. The identified processes refer to AR and IR, however the difficulty of students in establishing a general rule was recognized.

Task 2. This task, in Figure 2, involves a testing situation, in which a variable is expressed as a function of the coefficients of a linear system, causing different classifications in relation to its solution. Students are asked to analyze the relationship presented and identify the differences between the proposed cases. Furthermore, the relation corresponds to a fraction whose denominator is the determinant of the coefficient matrix of the system.

Figure 2. Task 2 - Question and testing performed by G1

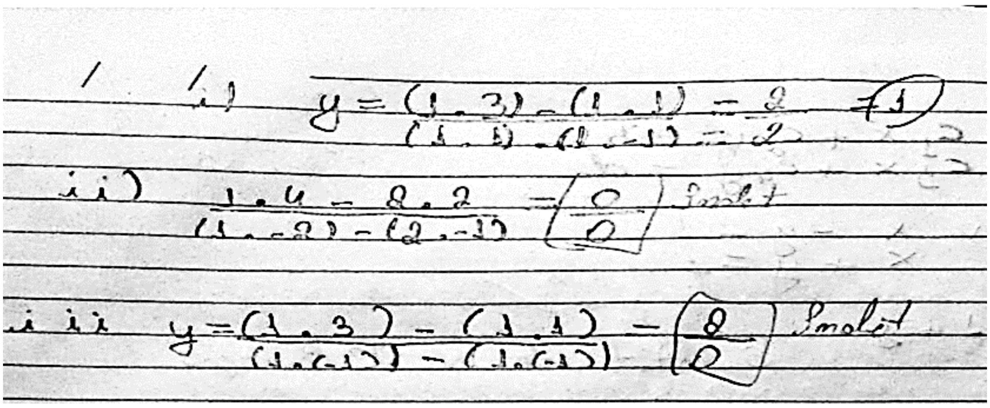
Consider the linear system below, where: c_1, c_2, c_3, c_4, b and d are real numbers (coefficients); x and y are variables to be determined.

$$\begin{cases} c_1x+c_2y=b \\ c_3x+c_4y=d \end{cases}$$

Present its resolution considering:

i) $c_1 = 1$; $c_2 = -1$; $c_3 = 1$; $c_4 = 1$; $b = 1$ and $d = 3$.
ii) $c_1 = 1$; $c_2 = -1$; $c_3 = 2$; $c_4 = -2$; $b = 2$ and $d = 4$.
iii) $c_1 = 1$; $c_2 = -1$; $c_3 = 1$; $c_4 = -1$; $b = 1$ and $d = 3$.

Regarding the solution, is the next expression true? $y=\dfrac{c_1d-c_3b}{c_1c_4-c_3c_2}$? Justify your answer. What differences do you identify in the solutions of cases *i*, *ii* and *iii*? Have the coefficient values influenced your answers? How? Do you identify a relationship between system coefficients and your solution? Present an algebraic expression that allows representing the relationship you have established.



From the data provided in the statement, first the students resorted to calculations, as an exploration tool, to identify patterns and to conjecture. The association established by testing the values and analyzing the linear system solution is evidenced in the following excerpts.

> G2: The first gave a possible and determined solution, the second gave a solution with infinite solutions, that is, possible and indeterminate, while the third gave an impossible solution (...) In the second system, the second equation is basically a multiple of the first. In the third, it is impossible that two equal equations have different results.

> G3: In i) we arrive at a unique solution, as no other value for the unknowns would satisfy the equation in the solution (...) In ii) we arrive at a possible and indeterminate solution, as the solutions are infinite, in case x would be depending of y or vice versa (...) In the last one, we reached the conclusion that this is an impossible system, as there is no solution to the equation.

In conclusion, students established the following implications, based on testing the values in the given expression (Figure 2): i) $y = 1$ => Possible and Determined System; ii) $y = 0/0$ => Possible and Undetermined System; and iii) $y = 2/0$ => Impossible System.

They performed tests based on the equation presented and, from that, classified the solutions. Therefore, the use of IR in this discovery through observation, testing particular examples, and identifying specificities is shown. The conclusion obtained does not have its validity formally verified, but allows an argument based on plausibility.

Task 3. The third task, in Figure 3, consists of direct verification of a possible solution for a linear system. Students are questioned about a false proposition, thus the question seeks to create opportunities to discuss the concept of a system solution, being able to understand solving algorithms, the underlying theory, which requires the formulation, testing and justification of conjectures.

During the resolution, the students first tested the validity of the proposition (possible solution). In principle, G3 stated that the vector [0, 1, 2] is a solution by evaluating the first two equations. Then, when asked by the teacher if the vector satisfies the system, they went back to testing the validity of all the equations and verified that the third equation is not satisfied. The other groups carried out this procedure directly, testing in all equations.

Figure 3. Task 3 - statement and answer presented by G1

About the linear system below,

$$\begin{cases} x+y+z=3 \\ 2x+3y+z=5 \\ x+y-2z=-5 \end{cases}$$

the vector [0, 1, 2] is solution? Why? Is there another solution? Which? Justify every answer.

It's not solution.
There is a divergence
in the third
equation.

> G2: The vector [0, 1, 2] is not a solution, as when verifying in x + y – 2z = -5, such condition is not valid because it is -4 =5, which is not true (...) The solution found was as follows [-4/3, 5/3, 8/3], as it satisfies the three equations.

In this case, regarding the found solution, [-4/3, 5/3, 8/3], aspects inherent to AR and DR can be considered. AR with regard to the solution, different from an initial proposition, with emphasis on the creative potential, but subject to the criteria established by rational argumentation and prior knowledge, the conjecture. And DR in relation to the justification established by the mode of assertion – Modus Ponens, which for any used propositions p and q, is characterized by the argument involving the premises (i) and (ii) in Figure 4. During the self-employment, there is an understanding about the validity of the conjunction, by admitting that e1, e2, e3, symbols used to represent the equations of the specific studied system, should be satisfied.

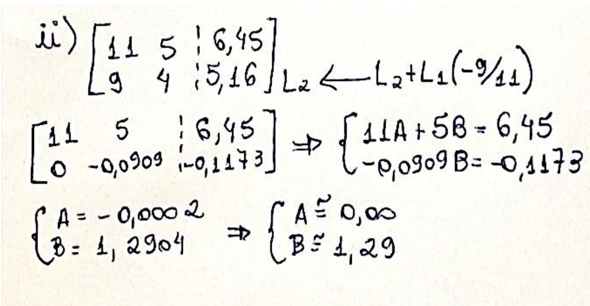
Figure 4. Task 3 – premises and symbols presented by G2

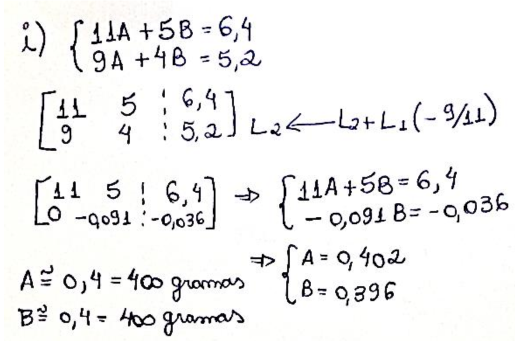
p → q (*i. first premise*)	(*e*1 and *e*2 and *e*3) → *s*
Modus Ponens p (*ii. second premise*)	*e*1 and *e*2 and *e*3 (*validity of the conjunction*)
q (*conclusion*)	*s*

Task 4. The fourth activity in Figure 5, involves studying the conditioning of systems. The context contemplates the elaboration of a mathematical model corresponding to a poorly conditioned linear system. The comparison between the solutions obtained in two situations, generated by a small numerical perturbation in the coefficient values, enables an investigation process. The solutions obtained by the students provide a rich scenario for discussion, due to the discrepancy of the results, between the disturbed systems (Figure 6), and between the groups (Figures 6 and 7).

Figure 5. Task 4 - Question and resolution by G2

On a scale that expresses the result in kg, with accuracy up to 100g, eleven packages of one product A and five packages of a product B were placed, and the weighing was 6.4kg. Then, two packages of product A and one of product B were removed and the weighing read was 5.2kg. This operation was repeated on another scale with precision up to 10g, so at the first weighing it was 6.45kg and in the second weighing it was 5.16kg. Considering the calculations with precision of three decimal digits, present the weights of products A and B for: *i)* the situation of the first scale; and *ii)* the situation of the second scale. Present the algebraic and graphic solution for the mathematical model corresponding to each case. What is possible to conclude about the results? Justify your answer. What is the weight of each product? Is there difference between the solutions obtained based on each of the scales? If so, what has led to this difference between solutions? Justify your answer.

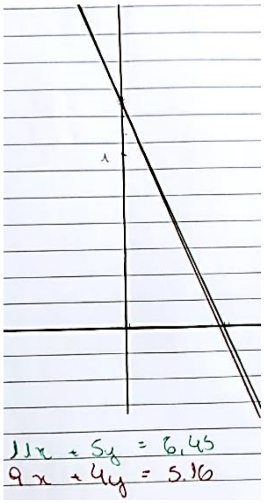




The results in each case were questioned. In principle, students observed that, as the products were the same, A and B, the results cannot be different in the situations generated by scales i) and ii). Then, they conjectured that this is specifically due to the difference in accuracy between the scales, "There are differences between the weights of the products due to the accuracy of each scale". They also emphasized that the linear system was the same, except for the variation in the second decimal place only in the vector that counts, "from 6.4 to 6.45 and from 5.2 to 5.16". The graphical representation of the solution (Figure 6), helped to understand what was happening that is, the difficulty in finding a specific solution for the linear system.

Figure 6. Answer presented by G1 to question 4

G1: The results obtained in the scales give different values for products **A** and **B**. In the first balance product **A** weighs **0.374** and product **B** weighs **0.457**, while in the second balance product **A** weighs **0.011** and product **B** weighs **1.267**. There are differences between the weights of the products due to the accuracy of each scale. In the graphs we can realize that the lines are on each other, the difference is that there is a minimum separation of the straights.



In the observations of this situation, the conjecture and the AR were highlighted. As the students did not understand how the same system could result in totally different solutions through a small numerical perturbation of the coefficients, their hypotheses emerged during the resolution, aiming to explain the unusual event that occurred, when they began to discuss and conjecture individually and in groups. However, it should be noted that a hypothesis resulting from AR admits the need for verification and objectivity supported by knowledge about a certain pre-established theory. In this case, although students showed signs of understanding about the mathematical content covered, they did not provide a consistent justification. At the end, in a plenary session involving all groups, the conditioning of a linear system, classification of your solution (SPD, SPI and SI) using as indicator parameters the determinant, the rank and the nullity, were addressed by the teacher. He retook the misinterpretations, to clarifying the concepts of a linear system, emphasizing that equivalence and proportionality are different situations, so the difficulties were solved when the teacher questioned the validity of an incorrect interpretation.

Task 5. The fifth activity explored follows the perspective of a linear system classification in relation to its solution. In particular, it seeks to provide opportunities for students to discuss based on the matrix representation of a system after using the Gaussian elimination method, to present their conclusions about it and their understanding of the underlying theory. Three situations are presented, as shown in Figure 7 and the following questions are proposed: "How do you relate the matrix resulting from the use of this numerical method with the solution of the linear system? What criteria do you use? Are they valid for any linear system, with m equations and n variables? Justify your answers".

Figure 7. Situations presented in task 5 and answers of G2

a) $\begin{cases} w+x+y=3 \\ 2w+3x+y=5 \\ w-x-2y=-5 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

b) $\begin{cases} w-x+2y=3 \\ w+2x-y=-3 \\ 2x-2y=1 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

c) $\begin{cases} w-x-y+2z=1 \\ 2w-2x-y+3z=3 \\ -w+x-y=-3 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a) In this case, we have in the last line of the system y=2, in this way we have an equation with one variable, so we can classify this system as DPS (determined possible system). That admits only a single solution.

b) In this system, we can see that the last line has null coefficients and independent term different than zero, z*0=5, so the system will be IS (impossible system), because any value that the variable will take on, will never be equal to the desired value.

c) In this system, the last line of the system results in null coefficients and null independent term, so we can classify it as PIS (possible indeterminate system), because there will be infinite values to satisfy the equation.

Unlike G2, G1 incorrectly interpreted the correlations represented in items b) and c) (Figure 7). The obstacle is related to the interpretation of the expressions: 0 = 5 and 0 = 0. So, with the teacher's intervention in the sense of questioning whether they were valid equalities and if involved variables, they concluded that: "*b) is impossible (SI), since 0 'times' y is different from 5; and c) depends on a variable for the solution (SPI)*". G3 also presented a similar answer, "*the first system has a single solution (y=2; x=3; w=2), the second system has no solution because 0 is different from 5, and the third system has infinite solutions, as it is free variable*".

With the exception of the first misinterpretation presented by G1, the reasoning process identified corresponds to justification, with DR, assuming logical coherence based on general principles. The conclusions are obtained directly through: p implies q or its contrapositive (if ~*q* so ~*p*). Students analyze d the last equation in each case and concluded on the following equivalences:

a) $-5z = -10$ <=> $y = 2$ *and, by values substitution, determine the unique solution of the linear system;*

b) $0 = 5$ <=> *contradiction, incompatible solution;*

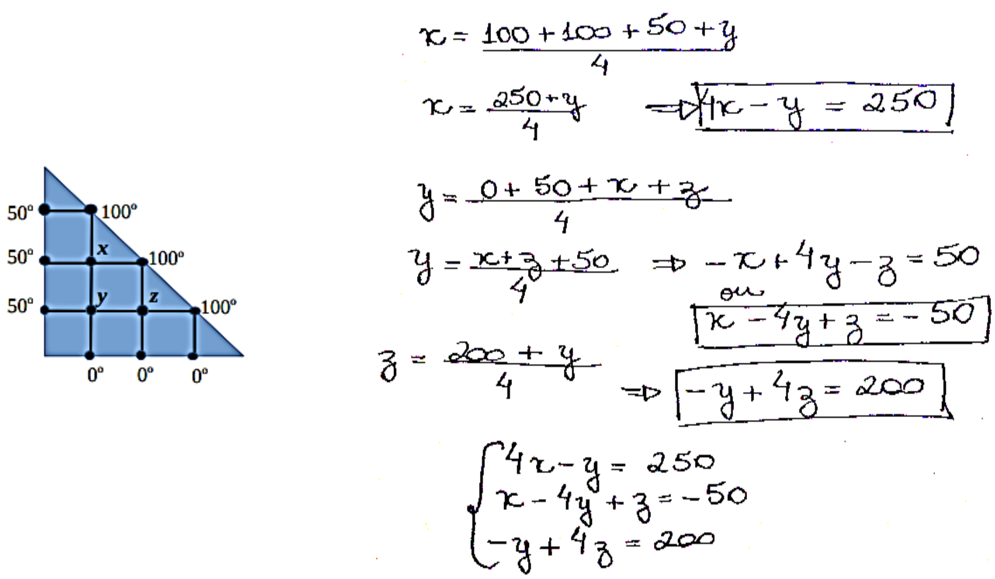
c) $0 = 0$ <=> *tautology, true sentence for any value attributed to variables, in the third equation.*

In the last case, c), they also identified the free variable, that is, the one that is not determined by the system and, therefore, can assume any real value, with the others being dependent on this value. However, when asked to express a generalization about the identified correlations, for any linear system, with m equations and n variables, they did not present an answer. And when asked to express a generalization about the identified correlations, for any linear system, with m equations and n variables, they did not present an answer. It was found that, although they had already studied concepts such as rank and nullity associated with the resolution of a linear system, they did not express processes involving such knowledge.

Task 6. The last activity investigated involves a context contemplating obtaining the temperature at different points on a triangular-shaped metal plaque. Students are asked to analyze and identify a mathematical model that allows completing some missing values in the representation of Figure 8.

Furthermore, it seeks to assist in understanding the resolution of a linear system using an iterative method, by identifying behavior patterns in which a variable can be expressed as a function of another(s). In this sense, the temperatures on the faces are provided, respectively, 0º, 50º and 100º C. Thus, a condition that the temperatures at the internal points, over time, will reach an 'equilibrium' resulting from the average of the temperatures at 'neighbouring' points is imposed. In other words, the unknown temperatures are obtained by averaging the temperatures of neighbouring points.

Figure 8. Resolution of task 6, by G2



Thus, the mathematical model corresponds to a linear system. During its development, there is initially the possibility of immediately employing the Jacobi or Gauss-Seidel iterative methods, according to a study carried out in the numerical calculus course. The variable that corresponds to the coefficient of the main diagonal of the coefficients matrix, is presented isolated in each one of the equations.

However, this iterative approach was not carried out directly by students. In principle, they sought to find values for x, y and z just by calculating the averages of constant values of the face temperatures. So, after the teacher's observation about the need to also consider the unknown temperatures of neighbours points, the students used the variables to calculate the averages (Figure 8).

In summary, solving involves the representation of equations corresponding to average temperatures and the generation of the equivalent linear system. None of the groups perceived that the mean temperature, as shown, corresponds to the standard for employing a basic iterative method. The answer obtained by the three groups, $x = 74{,}107^{o}$, $y = 46{,}429^{o}$ and $z = 61{,}607^{o}$ involves employing a direct method – Gaussian elimination. It appears that students had difficulty in interpreting the situation and associating it with a mathematical model, in particular, a linear system, as well as in solving it iteratively. In this case, the reasoning process identified is restricted to conjecture based on observation and identification of a possible solution to the problem, in accordance with the principles of AR.

5. Conclusions

In this study, carried out throughout an exploratory teaching experiment in a numerical calculus course of Mathematics degree, the PTs were challenged to perform tasks of a different nature from those they commonly use in higher education mathematics classes, that encourage them to experiment diverse MR processes aiming to develop their understanding of linear systems concepts and procedures. The work, developed in this context, provided opportunity to analyze the types and processes of MR that PTs use in carrying out research tasks involving linear systems, or the evidenced conceptual misunderstandings and reasoning difficulties, which allowed to identify how this context contributed to their learning in this thematic.

The overall results show that when PTs conceived solving a task as a process to get a result, they formulated diverse conjectures regarding the questions explored. In testing conjectures, they compared results or resorted to case experimentation, aspects that has occurred in other studies (Henriques & Ponte, 2014), although the generalization was less clear, and the argument to validate it was based on modes of affirmation not always coherent with the algebra propositions or the theorems proof.

This study shows evidence to conclude that students MR processes were mainly characterized by the conjecturing and justifying, associated with the use of AR, IR and DR, however they showed difficulty in some MR processes, in particular generalization and justification. Their IR was mainly characterized by testing particular examples and identifying specificities. Still, they revealed some facility in exploring and formulating specific conjectures, as well as in testing and comparing results to be concluded.

From the analysis of the results, the variety of representations that PTs were able to use in exploring the proposed research tasks, stands out as a significant aspect. All groups (G1, G2, G3) used verbal language, permeated by numerical and graphic representations in the elaboration of their reasoning, although the algebraic language was less evident in justifications, implying a greater degree of complexity as observed by Henriques and Ponte (2014).

Not surprisingly, with regard to the ability of MR, the teaching experiment carried out, of an exploratory nature, involving autonomous collaborative work and collective discussions, provided PTs an environment in which they shared and analyzed their reason and, when questioned, justified it by applying learned concepts and may identify their errors in solving tasks or reinforce their learning. Furthermore, those interactions and the orientations in a collaborative learning environment, helped students to realize new discovers and consolidate learning related to the specific content that resides: in understanding the meaning and nature of the solution of a linear system (tasks 1, 2, 3, 5); in the domain of solving processes and in concluding the results obtained (tasks 4, 5); and in the ability to solve problems represented in verbal language, based on the translation to algebraic language and on the interpretation of the solution according to the given conditions (task 4). There was an emphasis on the concept of equivalence between linear systems (task 1), for the use of parameters such as Determinant (task 2), Rank and Nullity (task 5) in the classification of the solution, as well as for the conditioning analysis (task 4). With this purpose, contributing to PTs learning, it was also relevant the teacher approach of concepts, properties and definitions in the final synthesis.

A limitation of this study is that it was developed in a virtual environment due to Covid-19 pandemic, that has been impacted the education including students' MR abilities, as argued by Nuramaliyah Ramadhany (2021), and the use of technological resources was not considered. Thus, further research is suggested, especially focus on higher education of PTs with an emphasis on linear systems linked to exploratory activities in classroom, development of MR ability and the use of resources such calculator or GeoGebra. Allowing to see the advantages of providing an effective learning environment to allow PTs to overcome their difficulties and achieve learning with understanding, as recognized by Fonseca & Henriques (2018).

Even so, as PTs solved the research tasks with success and several answers were of conceptual richness, this study thus reinforces the relevance of adopting this exploratory teaching approach, centered in their phases and involving mathematical tasks of an exploratory nature, in higher education. It was fundamental and showed a contribution to the evolution of the PTs mathematical reasoning, particularly their ability to conjecture and justify, based on the questions proposed during the work on tasks. And also proved to be appropriate for providing them opportunity to develop their knowledge of transversal curricular contents of linear systems such as its conditioning, considering the calculus of the determinant, classification in terms of its solution, considering the set and null parameters and their equivalences.

Finally, this study may assist in planning better educational contexts to improve undergraduate students' mathematical knowledge and reasoning, and the used activity can still provide PTs important knowledge for future teaching, particularly to promote students' MR.

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Mathematical reasoning in linear systems learning: a higher education exploratory teaching experiment with prospective teachers

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The importance of the theme of linear systems in higher education and the frequent difficulties that students show on concepts in this content, due to common procedural-dominated teaching practices, emphasize the need to develop their mathematical reasoning as an essential skill for their success in learning with understanding. To develop this students' reasoning skill and learning, to be successful, is highlighted the importance of using an exploratory teaching approach on pedagogical practices. However, in higher education, this approach on mathematical contents is not yet known or experienced by teachers in mathematical contents, including linear systems, especially in Brazil and in the context of pre-service teacher education, conditioning their future practices. Reflecting these concerns, it is considered relevant this study that carry out a teaching experiment with prospective mathematics teachers that were attending to the numerical calculus course of mathematics degree, in order to promote their mathematical reasoning and understanding of linear systems concepts and procedures, in an exploratory teaching context. This qualitative and interpretative study aims to analyse the mathematical reasoning that prospective teachers, attending the numerical calculus course of the 3rd year of Mathematics degree, use in solving research tasks involving linear systems, proposed throughout an exploratory teaching experiment and how this context contribute to their learning of linear systems. In particular, we addressed the following research questions: 1) what types and processes of mathematical reasoning do prospective teachers use in solving exploratory tasks involving linear systems? And what difficulties do they show in its use? 2) what are the learning carried out by the prospective teachers and how the teaching experiment contributes to their understanding of the linear systems concepts and procedures? Data collection includes participant observation of the teaching experiment classes, and prospective teachers written work of the proposed tasks.

The results show that prospective teachers have evolved positively in their understanding of the linear systems learning, and capacity of mathematical reasoning that were mainly characterized by the conjecturing and justifying. It is also evidenced that in this exploratory experiment, involving a collaborative learning environment, they share and analyze their reason and, when questioned, justify it by applying learned concepts and reinforce their learning.

The evidenced advantages of this experiment contribute to understand and reinforce the potential of adopting this exploratory teaching approach, integrating

mathematical reasoning and research tasks, to improve mathematics educational contexts in higher education, including preservice teacher education, helping mathematics prospective teachers to overcome their difficulties in learning of linear systems, and to develop their knowledge for teaching this mathematical topic and to promote mathematical reasoning in their future practices.