Task design in APOS Theory

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Diseño de tareas en la Teoría APOE

Resumen

Este trabajo discute el papel del diseño de tareas en la Teoría APOE. Se discute el papel que juega la descomposición genética en la teoría y en el diseño de tareas. Se muestra un ejemplo de descomposición genética para los conceptos de transformación matricial inversa y matriz inversa. Se proporcionan ejemplos que se diseñaron con dicha descomposición genética junto con una descripción de su relación con la misma a fin de dar una idea de cada una de las tareas y la construcción detallada y específica que tiene como objetivo. Se discute el papel de las tareas en el aula dado que la combinación del trabajo colaborativo de los estudiantes en secuencias de tareas y en la discusión en grupo constituyen la base de la Teoría APOE en la que se fundamenta su potencial para promover las construcciones necesarias para el aprendizaje profundo de los conceptos matemáticos.

Palabras clave. Diseño de tareas; Teoría APOE; transformación inversa; matriz inversa.

Task design in APOS Theory

Abstract

This paper discusses the role of task design in APOS Theory. The role played by the genetic decomposition in the theory and in task design is discussed. An example of a genetic decomposition for the concepts of inverse matrix transformation and inverse matrix is given. Tasks designed using this tool as a guide are exemplified as well as a description of their relationship to the genetic decomposition. In this way we provide insights about each task and the specific detailed construction it has as its aim. The role of the tasks in the classroom is discussed since the combination of collaborative work of students in sequences of tasks and in group discussions are the foundation of APOS Theory's potential to promote essential constructions needed for a deep learning of mathematical concepts.

Keywords. Task design; APOS Theory; inverse transformation; inverse matrix.

Projeto de tarefas na Teoría APOE

Resumo

Este artigo discute o papel do projeto de tarefas na teoria APOE. O papel da decomposição genética na teoria e no design das tarefas é discutido. Um exemplo de decomposição genética é mostrado para os conceitos de transformação inversa e matriz inversa. Exemplos que foram projetados com a referida decomposição genética são fornecidos juntamente com uma descrição de sua relação com ele, a fim de dar uma idéia de cada uma das tarefas e da construção detalhada e específica que ela tem como objetivo. O papel das tarefas em sala de aula é discutido, uma vez que a combinação do trabalho colaborativo dos estudantes em sequências de tarefas e em discussões em grupo constitui a base da Teoria APOE, que é isso que sustenta seu potencial de promover as construções necessárias para o aprendizagem profunda de conceitos matemáticos.

Conception de tâches dans la Théorie APOE

Résumé

Cet article discute le rôle de la conception de tâches dans la Théorie APOS. Le rôle de la décomposition génétique dans cette théorie et dans la conception des tâches est discuté. Un exemple de décomposition génétique est présenté pour les concepts de transformation inverse et matrice inverse. Des exemples conçus avec telle décomposition génétique sont fournis avec une description de leur relation avec celle-ci afin de donner une idée de chacune des tâches et de la construction détaillée et spécifique qu'elle a comme objectif. Le rôle des tâches en classe est discuté étant donné que la combinaison du travail collaboratif des élèves dans les séquences de tâches et dans les discussions en groupe constituent la base de la Théorie APOS, qui soutient son potentiel pour promouvoir les constructions nécessaires pour un apprentissage approfondie des concepts mathématiques.

1. Introduction

Task design is a common practice in mathematics education that can serve very different purposes. It can aim at building and/or consolidating knowledge and connections, engaging students in activities such as proving and conjecturing, revealing students' conceptions and misconceptions, even the most hidden ones (Sierpinska, 2004); creating a cognitive conflict to motivate progression in mathematical thinking (Aguilar & Oktaç, 2004); promoting argumentation (Schwarz & Linchevski, 2007); enhancing professional development of mathematics educators (Zaslavsky & Leikin, 2004); getting an idea about students' stance or intuitive responses about a particular topic and creating classroom discussions (Cline et al., 2013); creating a setting with the purpose of witnessing construction of knowledge taking place (Oktaç, 2019); promoting students' creativity (Lithner, 2017); helping teachers in organizing their class work (Lesh, et al., 2008), as well as assessing knowledge and abilities (Trgalová et al, 2014).

Despite the many purposes that task design serves in research and in teaching, as Sierpinska (2004) points out, the majority of studies do not comment about the strategies behind task design or variables involved in this process; therefore many aspects remain obscured and this is of little help to the rest of the community. An exception is research conducted from the viewpoint of theoretical frameworks in which task design plays a fundamental role, such as APOS Theory, realistic mathematics education (see Doorman, this issue) and theory of didactical situations (see García et al., this issue).

Sierpinska (2004) considers a *task* to be different from a *problem* in that a task involves only the minimum necessary effort in order to complete it (student's attitude), whereas a person who is engaged in solving a problem displays an independent intellectual's attitude. Although there might be some connotations attached to these terms, for the purposes of this paper, we will not distinguish between them. For us a mathematical task, problem situation or activity can involve varying degrees of depth of mathematical knowledge and reasoning, as well as connections with concepts in non-mathematical domains, depending on the research or teaching goals pursued. Our focus will be, from an APOS perspective, on the design principles that motivate the construction of specific concepts. Having clarified this point, we agree with Sierpinska in that "the design, analysis and empirical testing of mathematical tasks" is "one of the most important responsibilities of mathematics education" (p. 10).

The purpose of this paper is to describe the role and value of task design in APOS Theory, addressing the reasons behind the importance of task design in this theoretical framework, as well as the specific characteristics that tasks should hold in order to fulfill their purpose. In what follows, we first introduce APOS Theory briefly, focusing on the role of the genetic decomposition (GD) and the conception of task design in the theory. Next, we present some examples of tasks designed to guide students in the construction of *inverse matrix transformation* and *inverse matrix* concepts. We then present an example of the kind of interaction that can take place between the instructor and the students during a class discussion. We conclude with some reflections about the potential of task design based on APOS Theory and the role of the teacher in the class.

2. APOS Theory and the design of activities

APOS Theory is based on Piaget's epistemology about how knowledge grows (Arnon et al., 2014). It can be described in terms of its theoretical elements called structures, namely, Action, Process, Object, Schema and the mechanisms involved in moving from one structure to another. Although these structures are listed as stages in a hierarchical sequence, knowledge construction does not necessarily happen in a linear fashion. The learner can go back and forth between different structures or stay within one as long as it is needed. *Actions* are defined as transformations applied to an Object, such as manipulating it or using it to perform a calculation, following an algorithm or applying a memorized procedure; they are perceived by the learners as external in the sense that the learners cannot justify the steps that they follow or predict the result of their application. By reflecting on an Action (or a series of Actions) and repeating it, an Action can be interiorized into a *Process*, which means that the learner starts perceiving the Actions as internal, can omit steps and anticipate the result of their application. New Processes can be obtained from previously constructed ones by the mechanism of *coordination* when two Processes interact, or by the mechanism of *reversion* to construct an inverse Process. The need to transform a Process may lead the learner to perform or imagine performing Actions on it and hence to its *encapsulation* as an *Object*. Objects can be de-encapsulated to the Process from which they were originated. A *Schema* is a coherent construction composed of Actions, Processes, Objects and other previously constructed Schemas that are related to the same mathematical concept or topic. *Coherence* is a quality of an individual's Schema which implies that the learner is able to recognize when a problem situation falls within the scope of the Schema. The progression from one structure to another is dialectical in nature, which means that there can be passages back and forth between structures while knowledge is being constructed.

According to APOS Theory, a student's overall tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept depends on whether the student has constructed an Action, a Process or an Object conception of the concept or topic of interest or if a Schema is involved in the student's approach. Construction of knowledge develops in a spiral manner by applying new Actions on previously constructed Objects, progressively converting them into Processes, new Objects and Schemas.

A central component of APOS Theory is what is called a *genetic decomposition*, which is a hypothetical epistemological model describing the structures and mechanisms involved in constructing a mathematical concept or topic. As a theoretical conjecture, a GD predicts how a concept is constructed by a generic student; it must be experimentally tested and refined if necessary. There is no claim that there is a unique GD, that is, there is no pretension in the theory to describe how exactly a concept is constructed; several

GDs can exist (see for example Roa-Fuentes & Oktaç, 2010), but it is important that any proposed GD is tested experimentally with learners. The design of a GD involves reflection on the mathematics itself, on the history of the concept, on the experience of researchers and on research findings in mathematics education. The GD influences not only the design of research instruments and analysis of data, it is also essential as a guide in the design of teaching tasks and sequences.

Besides the description of the construction of mathematical knowledge, APOS Theory includes two methodological components that are also cyclical: A methodological cycle to test, refine and validate the proposed GD to make it compatible with research results obtained, and a teaching cycle known as the ACE cycle. This last cycle consists of the introduction of *activities* (A) for students to work collaboratively in small groups, whole *class group discussion* (C) and *exercises* (E) assigned as homework. Activities used in the teaching cycle need to be designed using the GD as a guide so that students have the opportunity to construct each one of the predicted structures and hence to learn what is being taught.

APOS Theory has also been used together with modeling approaches in order to design situations, taking into account a GD related to the concepts at stake. Student work on these situations is intermingled with activities consisting of several tasks that guide students' mental constructions and their modeling work (Trigueros, 2018).

In what follows, we describe a GD designed to construct the concept of inverse of a *matrix transformation* (a matrix transformation T_A *(v)* is a transformation such that T_A *(v) = Av*, where *A* is a matrix, *v* is a vector and *Av* is defined), and the concept of inverse of the aforementioned matrix *A*. As we will exemplify later, this GD guided the design of tasks in a study, with the intention to foster students' construction of the proposed structures in it.

3. Genetic decomposition of the concept *inverse of a matrix transformation*

We first mention the *conceptions* that we consider as being pre-requisites in order to construct the *concept* in question. However, before that, it might be necessary to make a distinction between these two notions: According to McDonald, Mathews and Strobel (2000) a conception "is intrapersonal (i.e., the individual's idea or understanding)" and a concept "is communal (i.e. a concept as agreed upon by mathematicians)" (p. 78).

3.1. Previous constructions

- Matrix and vectors as Objects as well as operations on matrices and vectors as Processes including the matrix- vector product.
- Vector transformation as a Process, which implies the possibility to identify for any vector *v* in the domain of a transformation *T*, a vector **w**=*T(***v***)* as its image under *T*.
- Matrix transformation as a Process, that can be seen as an important example of a vector transformation (Figueroa, Possani & Trigueros, 2018).
- Function of real variable as a Schema, which includes a function Process allowing the identification of an inverse image, as well as connections with other concepts such as set and variable.
- Systems of linear equations as a Process.
- Linear independence and dependence of sets of vectors as Processes.

3.2. Construction of the inverse of a matrix transformation

Since we are assuming that the students have constructed a function of real variable Schema as a prerequisite, they can now assimilate R^n as a set into this Schema as a valid domain/range, and hence think about finding the inverse of a transformation as a specific instance of an inverse function. However, if students have not constructed a strong function Schema, as it is commonly the case (Vidakovic, D. 1996; also see for example Baker, Cooley & Trigueros, 2000 and Cooley, Trigueros & Baker, 2007 for different levels of Schema construction), there might be other ways of imagining the inverse of a vector transformation. For example, the inverse function Process in the function of real variable Schema can be coordinated with the vector transformations Process through the recognition that the vector transformation as a correspondence rule may be reversed into a new Process that makes it possible to imagine the inverse of a vector transformation, and in particular, the inverse of a matrix transformation.

This Process is coordinated with the Process of the matrix representation of a system of equations through the recognition that finding the entries of the unknown matrix B associated to the inverse matrix transformation requires the solution of several systems of equations and if those systems need to have a unique solution, a necessary condition for the original transformation matrix is to be a square matrix.

This Process is repeated for different pairs of vectors *(v,w),* including the canonic vectors, and is coordinated with the linear independence Process by determining the conditions related to linear independence that have to be satisfied for matrix B to exist and be unique.

The need to perform the Action of comparing the results obtained in the previous Process makes it possible to encapsulate this Process into the Objects *inverse matrix* and *inverse matrix transformation*. Specifically, if the set of vectors **w** coincides with the set of canonic vectors, where the simultaneous solution of the systems $Av=e_1....\ Av=e_n$ is being searched, this construction is related to the Gauss- Jordan algorithm to find the inverse matrix.

4. Examples of task design for the case of an inverse problem

As part of a research project on teaching and learning linear algebra using models, the Blind source separation problem was used in an introductory linear algebra course at a private university in Mexico City. The modeling situation was expressed as follows:

Three important politicians will have a meeting to talk about a secret topic. You are spies and want to know what they talk about; so you install some microphones to record the sound in the room where they will meet. After they leave the room, you have four recordings and a map of the room indicating where the politicians were seated. What can we do to know the place where each of the politicians was seated?

Consistently with the ACE cycle, after some discussion in small groups of students about the task (which corresponds to the activity A in the cycle) including the importance of identifying each politician with their respective voices, students developed a functional model. With the aid of the teacher who asked questions such as *Why did you consider that the form of the room is important? And the type of voice?* and some experimentation done as a whole class discussion (which corresponds to C in the cycle), this model was extended to recognize the voice of each politician at an instant by means of a linear instantaneous mixture where voices are considered as pure tones. Students considered the distance between each source s_j and the observation point as a very important parameter and explained that the distance depended on the location of

each source (politician), with respect to the observation location (microphone); they referred to the resulting map as a *microphones' configuration* (Figure 1). Once a configuration was introduced as an example, students modeled the observation x_i, as a mixture in terms of a linear combination of the sources where each coefficient a_{ij} could be considered as a function of the distance, $d(ij)^{-2}$, for a given configuration, that is $x_i = a_{i_1}s_1 + a_{i_2}s_2 + ... + a_{im}s_{m,.}$ They also thought that sources and observations could be considered as vectors. Work was done with different configurations and different numbers of sources partly in class and partly as exercises (A and E in the cycle), after which they ended up with a general model of m sources and n observations; and the functional model was transformed into a matrix transformation model. Since observations were known, students needed to find the sources, so they worked on finding the inverse transformation and the inverse matrix to solve the problem of separating the voices of the politicians.
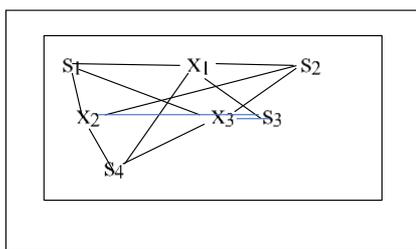

Figure 1. Microphone configuration with 4 sources (s_j) and 3 observations (x_i)

It took two weeks for students to arrive at this model through experimentation with sound and different configurations. Students observed that the inverse transformation solves the problem, since given a configuration of sources and observations, the observation vector x is known at an instant and that it is possible to represent the mixture with a matrix transformation. They also observed that the frequency of the sources could be associated to a tone corresponding to a voice.

Work on the task continued by another iteration of the ACE cycle. Students formed teams; their work on the modeling situation was accompanied by activities (A) based on the GD to help students to construct the inverse transformation and the inverse matrix. This was followed by a whole group discussion (C) where the teacher discussed students' ideas and formalized the mathematics involved in the activities. At the end of each class students worked on similar exercises at home (E) so they had new reflection opportunities.

Given this modeling context, we now present some examples of tasks that were prepared to guide students in the construction process. For each task we mention aspects that were taken into consideration in its design, as well as provide an *a priori* analysis in terms of the elements of APOS Theory. For an analysis of the results obtained by their application to students, see Vázquez Padilla (2017).

1. An entrance vector $s = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is mixed by a matrix $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ producing the exit vector $x = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$

   a) Is it possible to find a matrix transformation that reverts the effect of T_A? Explain.
   b) Suppose the new transformation is represented by a matrix B. What should be the size of B? How would you find it?

Question 1 was designed in terms of the modeling problem in order to stimulate students' thinking about the possibility to find an inverse transformation in specific situations. They are expected to solve the inverse problem by using their knowledge of matrix- vector product and its relation to systems of equations. It also aims to direct students' attention to more general situations by motivating them to reflect on properties of the solution sets of systems and on the characteristics of the matrix that should be satisfied in case it could be inverted. By working on several similar examples with varying conditions, students are expected to find the properties that would allow a matrix to be invertible. This is in line with the interiorization of Actions by repeating them and reflecting about them (Arnon et al., 2014).

In terms of the theory, Question 1 asks students to perform Actions on the given information to reflect on the relationship between matrix transformation and inverse matrix transformation, as well as their connection to systems of equations. Reflection on different similar tasks with varying conditions and different sizes of matrices can lead students to focus on the functional character of transformations and to construct a Process resulting from the coordination of the transformation Process and the system of equation Process. This coordination involves reflection on the possibility to find the inverse transformation, and, by comparing the results obtained with different matrices, students also construct a Process related to conditions that matrices should meet to ensure that the inverse matrix exists.

2a. If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and you know that $T_B \begin{pmatrix} 0 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, is it possible to find B? How?

2b. What would you need to know to make sure that you can find B?

2c. Compare your previous responses and answer the following:
    i) In which cases were you able to determine the inverse transformation T_B?
    ii) In which cases was it not possible?
    iii) Is there a relation between the vectors x and the possibility to find B?

These tasks ask students to perform specific Actions on the previously constructed Process with the intention to provide new opportunities of encapsulation of the inverse matrix transformation as an Object by specifying the properties shared by those examples when it is possible to find them. Examples of matrices of different sizes and including linearly independent or dependent columns and rows are used, both in activities during the lesson (A) and as exercises to work at home (E). After reflecting on the existence of the inverse transformation and the related matrix, tasks such as those exemplified here have the goal of promoting students' reflection in a more general context, so that they can think about the existence conditions of an inverse matrix. The use of matrices of various sizes contributes to such reflection. It is expected that after completing the tasks students would have a first general idea of which properties a matrix associated to a matrix transformation should satisfy in order to be invertible.

These tasks also involve the coordination of the Process of matrix transformation and that of systems of equations but aim at focusing students' reflection on the conditions needed to find the inverse matrix and the inverse transformation. The Actions of interpreting the solution set and adding the conditions needed for it to be unique help the encapsulation of the Process resulting from the coordination, into an Object *inverse matrix* through the consideration of its properties. Repeating this task by varying the vectors involved using linearly independent or dependent vectors, including one zero vector or vectors that correspond to the column of the inverse matrix, promote opportunities to reflect on the construction of the inverse transformation Process

resulting from the preceding coordination. At the same time its coordination with the Process of linear independence and dependence through comparison of results obtained by using different matrices gives rise to a new Process where the existence of the inverse transformation can be determined without the need to perform all the involved calculations. Opportunities to encapsulate the Processes of inverse matrix transformation and inverse matrix as Objects are also provided.

3. Suppose you need to find the inverse of the matrix $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$
   You have the following information options:
   Option 1: $A(1, 1) = (1, -2)$, $A(0, 1) = (1, -3)$
   Option 2: $A(3, 5) = (1, 0)$, $A(1, 2) = (0, 1)$
   a) Which one would you select to find the inverse matrix and the inverse transformation? Why?
   b) Is there a relation between the entrance data and the matrix A's inverse? Explain.
   c) Find the inverse matrix for A, if possible, using your responses to the previous questions. If it is not possible, explain why.

This and other similar tasks for different matrix transformations (A, E) have the aim to provide students with opportunities to think, before making any calculations, about what the result can be and how easy it would be to find the inverse of a given matrix. Also of interest is to motivate them to look for the inverse of a matrix, given information about the result of the application of a transformation to different vectors, as well as to compare results leading them to discover that if the inverse matrix exists, it is independent of the data used. Comparison of applying the same procedure with different data has the goal of making students aware of the differences involved in the procedure when vectors selected are easier to handle and to bring forward the role of canonic vectors when finding the inverse of a matrix.

In terms of APOS Theory, reflecting on operations without actually carrying them out motivates the interiorization of Actions, whereas thinking about the properties that should be satisfied helps with the encapsulation of the related Process. The task is repeated for matrices of different sizes and with different properties. The goal is for students to compare, in terms of efficiency, the results of using different sets of information as well as to reflect on the existence of the inverse matrix by examining and comparing the properties of different matrices. Here we are using the word *efficiency* as related to the effort that goes into obtaining the same result by different methods; it does not belong to the terminology of APOS Theory, rather it emerges in the context of modeling activities. Comparison also favors the encapsulation of the Processes of inverse matrix transformation and the inverse of a matrix. It should be underlined that the choice and ordering of tasks is key in APOS Theory.

4. Consider the following matrix transformation $T_A(s) = \begin{pmatrix} 2 & 1 & -4 \\ 4 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix} s$

   By using what you have found before, find the inverse transformation. Analyze what you did.
   a) Is it possible to represent the transformation as a system of equations for each canonical vector? Find the representations if possible.
   b) Solve the systems using the Gauss-Jordan method; remember that it is possible to solve several systems simultaneously. Is there a relation between the systems' solution and the inverse matrix? How do your results compare with what you did before?

The objective of this task, and other similar ones using for example the matrix $\begin{pmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}$ is to continue giving opportunities to students to reflect on the existence of an inverse transformation by using those properties they had found before and to discover that what they have done can be summarized in an efficient procedure to find the inverse of a matrix transformation and of a matrix. Many textbooks present this technique without clearly explaining students why it works. The previous tasks have prepared the students to give meaning to this procedure and to understand it in terms of the properties of the canonical vectors.

The purpose of this task, in line with the GD, is to provide students with an opportunity to make use of what they have constructed so far in order to develop a technique to calculate the inverse matrix and the inverse transformation, with understanding. This way Actions can be interiorized into Processes, allowing students to have control over the method, helping them to understand how and why it works. This technique is usually presented in textbooks to find the inverse matrix without any relation to transformations and without explaining why this procedure works.

5. Write down a summary about the possibility of existence of the inverse matrix transformation and about the conditions needed for its existence as well as the existence of the inverse matrix.

The purpose of this task is to provide a new opportunity for students to reflect about all the constructions that they have made so far and contribute to the construction of new connections if they had not had the chance before.

Work on the tasks presented above was intermingled with similar tasks referring to the modeling situation, so that at the end, the students would come up with responses to the original problem situation, with an understanding of how other similar inverse problems can be solved.

It is important to underline that all the work done by students during the Activities phase of the ACE cycle are discussed at each turn of the cycle in a whole group discussion. During these sessions the teacher provides new opportunities for students to reflect on what they have been able to do and also on what they have not been able to do, in a way that aims at promoting new constructions. It is during these discussions that students are encouraged by the teacher to formalize their findings which in turn leads to the consolidation of the constructions shown by students so far.

These tasks illustrate the value of a genetic decomposition as a tool to guide a careful and detailed design of sets of tasks that promote the construction of each of the predicted structures and mechanisms. Designing tasks to promote all the constructions described in a genetic decomposition represents a challenge to researchers and teachers; it is not an easy endeavor, but many studies using APOS Theory have found that once achieved, the tasks designed are not only original, but also powerful in helping students to construct the intended concepts and methods so they learn with meaning the intended mathematics.

When modeling activities are introduced together with activities designed with the GD, the modeling tasks are intermingled with tasks that help students reflect on what they have constructed and its relation to the modeling situation. This strategy promotes the evolution of work on the modeling situation, and helps the students to realize the role played by hypothesis and the limitations of the model or models developed.

In accordance with APOS Theory, a GD needs to be validated in terms of students' constructions. The one that we present in this paper was validated by means of analysis of student work on the activities (Vázquez Padilla, 2017), although no interviews were performed. In general, in APOS-related research, individual interviews are the preferred method of data collection, since they offer depth of observation and allow the interviewer to follow up with questions if needed. In a modeling environment however, it is important to let students work together and observe them while they are working on specially-designed tasks together. From the viewpoint of APOS Theory, if a GD does not coincide with the preliminary theoretical analysis it needs to be refined and new tasks would need to be developed in order to consider the new constructions included, and this GD as a whole would need to be experimentally tested.

5. The role of the teacher: An example

All the activities discussed in the previous section were worked out by the students in teams of 3 or 4 students. The teacher interrupted students' work when they finished a set of activities related to each question shown before, and sometimes when she felt the need to discuss students' specific questions and doubts. This type of interventions is fundamental in what is designed as class discussion (C) in the ACE cycle. In this section we provide an example of the kind of questions and comments of the teacher when discussing question 2 (T stands for teacher, and S followed by a number stands for a specific student).

T: What did you observe when solving part (a)?

S1:   We found that it was not possible because when you solve the system there are many possibilities for two of the four variables.

T: And what does that mean?

S2:   When you have parameters you can give them any value you want.

S1:   But then it would not be necessarily consistent, you give different values to the parameters and you find different matrices, but you need the same for all the given data,

T: Is the matrix corresponding to the inverse transformation unique?

S2:   Yes, now I see, then it is not possible.

T: Does someone have a different response or a comment to make? (No response). Then what about part b?

S3:   You need more information, another source vector and its mixing result. It has to be different.

T: What would you expect to happen if I give you a source vector that is a multiple of the one you had?

S3:   It will not work. Again we would have many solutions. Ah, I see what you meant. The new information vectors have to be linearly independent so we can find matrix B and the resulting vectors from the transformation also have to be independent so the system that we obtain will have a unique solution.

T: Now how do you interpret matrix B?

S2:   You know the matrix transformation corresponding to matrix B applied to an s gives the vector x; then if you multiply B by s you have to find the given x.

T: That is fine, but if you compare what you are doing in part (a) with what you are doing in part (b), what is the difference?

S2:   I see that, well matrix B is the matrix associated to the inverse of the transformation.

T: What can we say now after doing this activity?

S3:   Comparing all, we know now that for a matrix transformation its inverse transformation is unique and so is the matrix associated to it. And then, comparing all the systems we did, we found that for the inverse matrix to exist the matrix associated to the original transformation needs to be squared and its rows and columns must be linearly independent.

6. Discussion and conclusions

The analysis of the examples presented shows the close connection between a GD and the design of problem situations. Each task focuses on details that many times are not considered when the instructor or the designer assumes a broader view of the topic to teach. By having the opportunity to center on each construction when working with sequences consisting in this type of tasks, students are given many opportunities to reflect on very fine- grained details of mathematical concepts from different points of view. Students are also given opportunities to discuss the role played by each task in itself and in terms of a sequence, as well as the opportunity to develop their strategies at different moments of the ACE cycle. They are also encouraged to relate all those constructions and build a deeper understanding of the concepts involved.

There might be a doubt as to whether the use of ACE cycle and the designed activities can constrain the instructors' freedom to use other different tasks that had not been previously developed. However, this is not the case. In each cycle the instructor can ask students questions considered to be important, with the aim of helping students constructing knowledge while they work in teams and can use the ACE cycle phases as considered necessary depending on the appreciation of the needs of the group in terms of where the students stand as for the construction of their knowledge about the topics in question. He or she can also introduce new questions that motivate revisiting of the previously worked tasks, with the intention to help students to reconsider or refine their approaches, or to keep them reflecting on what they have done, always under the guidance of the proposed sequence. These opportunities enrich the instructors' work.

Task design in APOS Theory is also useful in detecting and explaining students' ways of thinking, their difficulties and the relations between different constructions they can or cannot make. There are many examples that show its potential and success in explaining learning phenomena (i.e. Weller et al., 2003; Roa-Fuentes & Oktaç, 2012, Martínez-Planell & Trigueros, 2019). There is also a wide spectrum of tasks designed and tested to teach calculus, linear and abstract algebra, and other areas of mathematics.

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Task design in APOS Theory

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Extended abstract

This paper discusses the role of task design in APOS (Action–Process–Object –Schema) theory in relation to student learning of mathematical concepts. The theoretical framework is explained briefly, followed by an explanation of the role played by the genetic decomposition in the theory and in task design. An example of a genetic decomposition for the concepts of inverse matrix transformation and inverse matrix is given. Tasks designed using this research tool as a guide are exemplified, together with a description of their relationship to the genetic decomposition. This provides insights about each task and the mental constructions that it has as its aim. A detailed preliminary analysis from the viewpoint of APOS Theory points out to the connection between the mental structures and mechanisms predicted by the genetic decomposition, and the design of the components of the task sequence. Specific features that each task holds in order to fulfill its purpose are mentioned, as well as the role that it plays within the sequence. An overarching modeling context offers opportunities in terms of linking mathematical ideas to an engineering problem adapted to an introductory linear algebra course, of motivation for solving it, raising the interest of students for the mathematical concepts in question as well as preparing them for making connections with more advanced notions. Designing task sequences within the methodological cycle of APOS Theory ensures that the tasks are original and powerful in helping students learn with meaning the intended mathematics. The role that tasks play in the class is also discussed, as the combination of collaborative work of students in sequences of tasks and in group discussions form the foundation of APOS Theory's potential to promote essential constructions needed for a deep learning of mathematical notions. Considerations about strategies to be employed in the classroom by instructors are included.