

Impact of theoretical perspectives on the design of mathematical modelling tasks

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Abstract

In this paper, we discuss how the adoption of a particular theoretical framework affects task design in the research field of modelling and applications. With this purpose, we start by referring to the existence of different reference epistemological models about mathematical modelling to analyse better the consequences they have for decision making concerning designing modelling tasks and their implementation. In particular, we present the analysis of three case studies, which have been selected as representatives of different theoretical perspectives to modelling. We discuss the impact of the chosen reference epistemological model on the task design process of mathematical modelling and the local ecologies suited for their implementation.

Keywords. Mathematical modelling; reference epistemological models; task-design; modelling practices; didactic ecology.

Impacto del enfoque teórico en el diseño de tareas de modelización matemática

Resumen

En este trabajo discutimos cómo la adopción de un marco teórico concreto incide en el diseño de tareas en el ámbito de investigación de modelización y aplicaciones. Con este objetivo, comenzamos refiriéndonos a la existencia de distintos modelos epistemológicos de referencia sobre la modelización matemática para analizar sus consecuencias en la toma de decisiones sobre el diseño e implementación de tareas de modelización. En particular, presentamos el análisis de tres estudios de caso, que han sido seleccionados como representantes de diferentes enfoques teóricos sobre modelización matemática. En base a estos, discutimos el impacto del modelo epistemológico de referencia elegido en el proceso de diseño de tareas sobre modelización y las ecologías locales planeadas para su implementación.

Palabras clave. Modelización matemática; modelos epistemológicos de referencia; diseño de tareas; prácticas de modelización; ecología didáctica.

1. Introduction

Applications of mathematics and mathematical modelling have become core elements of mathematics education across the world during the last decades. Research in mathematics education has made great efforts to study the teaching and learning of mathematical modelling, often linked to the justification of and motivation for learning mathematics (Blum & Niss, 1991; Galbraith et al., 2007). Modelling has been integrated into curricula for good reasons, as Burkhardt (2018, p. 74) states:

“The importance of mathematical modelling in the school curriculum is clear. It both demonstrates the widespread applicability of mathematics and enhances mathematical understanding through inquiry. It serves as a powerful corrective to those who view mathematics as a set of discrete facts and procedures to be taught and learned.”

The research community on modelling and applications has sought to clarify and connect diverse frameworks for designing and analysing mathematical modelling activities (Kaiser & Sriraman, 2006; Cai et al., 2014). Lesh and Fennewald (2013) point out that one major challenge is the “conceptual fuzziness” about what counts as a

modelling activity. When the teaching and learning of modelling become an object of study, researchers elaborate on different conceptions of mathematical modelling. These conceptualisations take different shapes depending on the perspective adopted. The focus of our paper is to illustrate a variety of theoretical frameworks in use and the impact of choosing a particular framework on task design in modelling.

To analyse and compare frameworks of the research on modelling and applications, we draw on tools from the anthropological theory of the didactic (ATD) and the notion of reference epistemological model. As argued by Bosch and Gascón (2006), when a research problem in mathematics education involves a specific mathematical content or process such as mathematical modelling, researchers build a specific reference epistemological model, that is, a vision of mathematical modelling. Reference epistemological models are also a tool to avoid assuming—without questioning—the prevailing way of conceiving modelling activities. These models appear as alternative descriptions of the knowledge at stake, e.g. modelling, as it is proposed to be taught and learned in school institutions.

Barquero, Bosch and Gascón (2019) have argued that different theoretical approaches in mathematics education lead to define different reference epistemological models that affect the research problems addressed and the empirical reality considered. Less explored is the analysis of the impact different reference epistemological models have on the principles guiding the design of modelling practices. This is the focus of our discussion that deals with the following questions:

What reference epistemological model does a particular research framework build to approach the teaching and learning of mathematical modelling? What principles guide the design of modelling practices in a given theoretical framework? What impact do the different reference epistemological models have on the design principles considered?

We approach these questions by considering three theoretical perspectives: the “modelling cycle and competencies approach”, the “models and modelling perspective” and the “anthropological theory of the didactic”, which will be described in the next section. Jessen, Hoff-Kjeldsen and Winsløw (2015) analysed and compared two teaching implementations with modelling, one from the modelling cycle and competencies approach and one from the anthropological theory of the didactic. In this paper, we go beyond differences in classroom realisations, to analyse what defines modelling in the three perspectives and the consequences for design principles. We start by presenting the reference epistemological models elaborated by the three frameworks. Then, we investigate relations between these reference epistemological models and the corresponding design principles for modelling proposed in three studies, one in each framework. The three frameworks do not cover the research on “modelling and applications”, nor the three studies represent all design principles embedded in each of them. Still, they all display the variety of meanings of the term ‘modelling’ and why we need to be cautious with not assuming general consensus. The case studies have been selected because they represent the main ideas of each framework and include examples that explicate principles for modelling task design. We conclude with some results from the comparative analysis across cases.

2. Emergence and development of frameworks for modelling and principles for designing modelling tasks

As mentioned before, when the teaching and learning of mathematical modelling become an object of study for the research community in mathematics education,

researchers elaborate their conception of what modelling is and how to approach it. We refer to it as a *reference epistemological model* (Bosch & Gascón, 2006), which appear as alternative descriptions of the knowledge at stake in teaching and learning processes, built from research frameworks. In this paper, we are particularly interested in the impact they have on the design of modelling practices.

For analysing the frameworks' impact on the design of mathematical modelling, we may also refer to Kieran, Doorman and Ohtani (2015) who provide an overview of the research in mathematics education on task design. These authors distinguish “grand theoretical frames”, “intermediate-level frames” and “domain-specific frames.” Grand theoretical frames present theories about learning at a general level, inside and outside of formal educational settings. Constructivist learning theories, adapted to mathematics education, are given as examples of this grand theoretical frames. Intermediate-level frames present “the complex interactions between task, teacher, teaching methods, educational environment, mathematical knowledge, and learning so that the purposes and implications for task design are always understood within the total structure of practice” (Kieran et al., 2015, p. 5). Realistic mathematics education, the theory of didactic situations and the ATD are considered examples of such intermediate frameworks. Domain-specific frames often focus on particular areas of mathematical content knowledge (geometry, analysis, etc.) or specific mathematical processes (e.g., conjecturing, proving or modelling) and may not be easily generalizable across other mathematical topics. One feature of the domain-specific level is that the designed tasks usually follow the purpose of supporting students in the learning of specific content or process knowledge. This way of conceptualizing design frames is used as a backdrop for examining principles for task design in mathematics education research. We adopt this distinction between levels of frameworks, to discuss the principles for task design in the research field of mathematical modelling.

In what follows, we start by introducing the reference epistemological model (REM) for mathematical modelling built upon the three theoretical frameworks here considered. It will be followed by a general description of the methodology followed for the analysis of our selected case studies.

2.1. Different reference epistemological models on mathematical modelling

As indicated in English et al. (2015, p.384), one of the prevailing approaches to modelling is summarised in a schematically and idealised way about how the modelling process connects the extra-mathematical and mathematical worlds through the so-called *modelling cycles*. We refer to this first approach as “modelling cycle and competencies approach.” It introduces a modelling cycle as a sequence of sub-processes starting in a real-world situation, moving into the world of mathematics, where mathematical models and results are elaborated, and then validated and reinterpreted by moving back to the real-world situation. One of the modelling cycles, proposed by Blum and Leiß (2007), has been widely used. As shown in Figure 1, it is described through seven phases (understanding/constructing; simplifying/structuring; mathematising; working mathematically; interpreting; validating; exposing), assuming that all individuals more or less proceed through these phases during modelling.

The modelling cycle (with its variations) seems particularly helpful as a reference epistemological model to analyse modelling processes followed by students and teachers under a cognitive perspective (Borromeo-Ferri, 2007). It is also used to focus on modelling competencies, which are often directly defined by referring to the

modelling cycle, as the skills to perform the modelling sub-processes (Kaiser, 2007). Greefrath (2020, this issue) presents an overview of developments within the approach of modelling cycles and competencies in Germany, showing how modelling cycle(s) and competencies provide an instrument for diagnostic and assessment purposes, and a foundation for intervention in schools and teacher education.

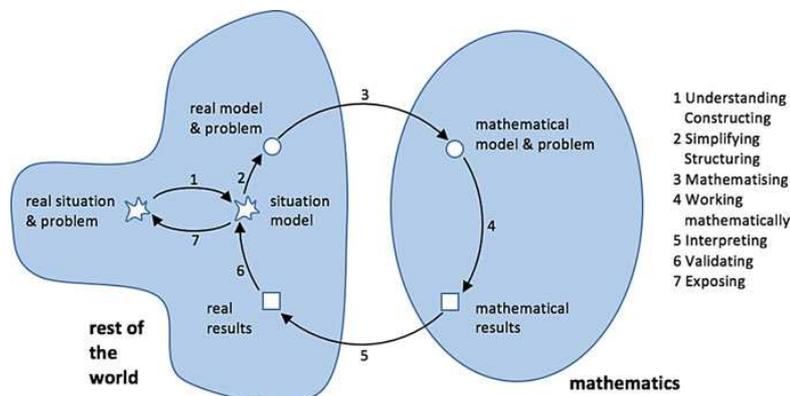


Figure 1. Description of the modelling cycle (Blum & Leiß, 2007, p. 225)

The second approach we consider is the “models and modelling perspective” typically addressed through the *model-eliciting activities* (Lesh et al., 2000). As stressed by Czocher (2017), this framework represents a more holistic approach to mathematical modelling by the way it supports the learning of mathematical content knowledge. Designs proposed in this framework consider modelling activities as particularly suitable for promoting student engagement and for its potential to support teachers’ and students’ engagement with modelling practices (Lesh et al., 2003).

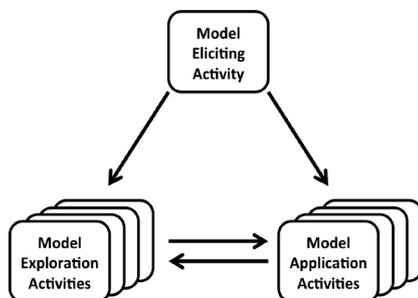


Figure 2. Structure of a model development sequence (Ärlebäck & Doerr, 2015)

Modelling activities are here described and considered in terms of *model development sequences*. As explained in Ärlebäck and Doerr (2015, 2018), such a model development sequence starts with a model eliciting activity (MEA), followed by model exploration activities (MXA) and model application activities (MMA) (Lesh et al., 2003; Doerr & English, 2003) (see Figure 2). MEAs are designed to confront the students with the need to construct a model to make sense of the problem situation. The MXA focus on the underlying structure of the elicited model, when models are developed accordingly to the strengths and ways to use them productively. Then, these models are adapted and applied (MAA) to new situations that appear beyond their initial context. As Doerr and English (2003) explain, when students work through a model development sequence, they engage realistic problems, that reveal the multiple ways they address the tasks. Students also engage in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students and participating in teacher-led class discussions.

A well-established line of research has worked on the design of model eliciting activities in multiple contexts with learners from primary school to university. Researchers in this line distinguish among six common principles for designing MEAs, which have been guiding the design of modelling activities over the last decades. These design principles will be used when we analyse the second case study.

Since the first developments of the ATD, mathematical modelling is linked to the notion of mathematical activity. It is assumed that doing mathematics mainly consists in the activity of producing, transforming, interpreting and developing mathematical models (García et al., 2006). The relations between mathematical modelling and the construction of mathematical concepts is approached through the notion of *praxeology*, which is the primary ATD tool to describe knowledge and activities in institutional settings (Chevallard, 1999). The notion of praxeology links the conceptual and procedural aspects of human activities by including, as inseparable entities, the *praxis*, made of types of tasks and techniques to solve them, and the *logos*, made of discourses and theoretical tools to describe, explain, justify and nourish the *praxis*. Like any other human activity, modelling a given situation to obtain new information or knowledge about it can be described in terms of praxeologies. From the starting point there is a task we want to solve, we use a technique to produce a model of the situation or system underpinning the task, we sustain this praxis by notions, tools, and justifications provided by the theory (or logos). Moreover, once a given system has been modelled, a new praxeology can be developed by integrating the model produced into new techniques to solve new tasks within a more developed logos.

Modelling is thus a process of constructing a sequence of mathematical praxeologies that become progressively broader and more complex. Furthermore, in this process, modelling does not appear as a closed cycle with a beginning and an end, but as a continuous process enhanced by the raising of new questions. It is also a recursive process since each model proposed can, in turn, be questioned and become a system for a new modelling process. This enables the connection and coordination of mathematical models into broader and more complete knowledge organisations.

Since the last 15 years, several researchers in the ATD have been working with the proposal of the *study and research paths* (Chevallard, 2006 and 2015) as instructional devices for the teaching of mathematical modelling at different school levels. The design principles for modelling are then embedded into the design principles of study and research paths (Bosch, 2018; García et al., 2019).

2.2. Case selection and methodology of analysis

We focus on three case studies related to different theoretical approaches to modelling. The cases are analysed through the research papers, selected according to the criteria of representing one of the theoretical approaches considered and including information about the task design process. Table 1 summarises the selection of papers.

Table 1. *Selection of papers for each theoretical approach*

Case	Theoretical approach	Selection of research papers
1	Modelling cycle and competencies	Maaß (2010), Kaiser & Schwarz (2006)
2	Models and modelling perspective	Ärlebäck & Doerr (2015, 2018)
3	Anthropological theory of the didactic	Fonseca, Gascón & Lucas (2014), Lucas (2015)

Our methodology is based on a qualitative analysis of the selected papers. We can distinguish three levels according to what we analyse in the case studies and how we report the results. First, we describe the mathematical modelling activity at the core of the design, and the theoretical approach adopted. Moreover, we inform about the teaching device/s planned and the school conditions for the implementation. Second, focusing on the task design for the modelling activity, we underline the design principles considered by each approach and exemplify how these principles are taken into consideration in the particular example(s) illustrated in the paper(s). Third, we are interested in the advances and discussion the different cases present about the “local ecologies” for modelling. We use this term to refer to the local conditions that are created for the implementation of the modelling teaching and learning practices. When we use the term ecology, we refer to the set of conditions that favour, and constraints that hinder the implementation of modelling practices. We align with what Czocher (2017) named “learning ecologies” or Borba and Wake (in Cai et al., 2014) called “classroom ecologies” but, in our case, ecology includes conditions and constraints beyond the individual or classroom reality (Barquero, Bosch & Gascón, 2019).

3. Case studies for the impact of modelling frameworks on the design of modelling

3.1. Case study 1: Designing tasks in the modelling cycle approach

Concerning the modelling cycle and competencies approach, we had difficulties in selecting research papers since most of them do not define the principles for modelling task design. What seems clear is that the definition of the sub-processes in the modelling cycle and of the modelling sub-competencies is often the base for the design of modelling tasks for students and teachers. Maaß (2010) develops a classification scheme for modelling tasks. This schema provides an overview of the features of modelling tasks and offers guidance in task design. Some sub-categories are considered to classify modelling tasks, such as which parts of the modelling process have to be done, what type of data is given, what the relation to reality is, what type of models is used, what the level of openness in the question is, and what the cognitive requirements are. For instance, the first category, called “Focus of the modelling activity”, refers to which parts of the modelling process are fostered through the design of the modelling task. The categories for the analysis are expressed accordingly to the steps in the modelling process “carrying out the whole process of the modelling cycle”, or covering parts of it “understanding the situation” or “mathematising”, or “working within mathematics.” In that paper, the classification scheme is used to design and to analyse modelling tasks that emerged at the forefront of a research project. The target group is low-achieving 12-year-old students. The aim was to design consecutive teaching units that help to develop competencies successively in the carrying out of single steps of the modelling process. The following examples are modelling tasks design in the frame of this project. Table 2 summarises the main traits discussed in the paper according to the categories in the classification schema.

[Task 1] Imagine you want to paint your room. Which colour would you choose? How much paint will you need if 1 litre is sufficient for 6–8 m²? (p. 307)

[Task 2] Elias lives in Weingarten, a part of Freiburg. The tram stops in the street where he lives is called “Rohrgraben”. Today he wants to watch a football match with the SC Freiburg in the stadium, which is located in Littenweiler, line 1, tram stop “Hasemannstraße”. Before he goes there, he wants to pick up his friend, who lives in

“Hornusstraße” (line 5) in Zähringen. Describe the route, Elias has to follow when going from his house to the stadium. How often will he change trams? (p. 307)

Table 2. *Design principles according to the classification scheme for modelling tasks*

Design principles	Discussed with the designed tasks
<p><i>Focus of the modelling activity</i> Do students carry out the whole modelling process or only some steps?</p>	Design of task(s) accordingly to the step(s) of the modelling process to help students develop modelling competencies.
<p><i>Type of data given</i> Which data is provided?</p>	Tasks contain more data than needed (superfluous data) or missing data, and students are expected to gather additional information. In Task 1, e.g., some data are missing and relevant data may be identified.
<p><i>Context and relation to reality</i> What is the nature of the context's relations to reality?</p>	Tasks are discussed according to the authenticity of the context and the data given. and questions posed to students. In Task 1, e.g., it is considered students' interest in the reality related.
<p><i>Type of models and representation</i> Which type of model is used? Which type of representation is chosen?</p>	The types of models that appear are classified as descriptive or normative. About Task 1 the model expected to appear is to sketch a formula; or in Task 2 a graphical model by drawing the solution.
<p><i>Cognitive demand</i> What are the cognitive demands with respect to certain competencies?</p>	The cognitive demand is discussed in terms of complexity of the situation: how demanding is the transference from reality to mathematics, the complexity of working in the mathematical world, and on the choice of terminology to express the mathematical and extra-mathematical information. Again, this discussion is linked to the sub-processes on the modelling cycle required through the task.
<p><i>Mathematical frame</i> Which mathematical frame does the task have?</p>	Tasks are placed accordingly to the mathematical curriculum and the school level. For instance, Task 2 is analysed in terms of the likely content to appear: theory of graphs, meaning of knots and edges, and placed in lower secondary, according to curricula.

By selecting these examples, we do not mean that all modelling tasks are discussed in these terms in the modelling cycle and competencies approach. Vorhölter et al. (2019) describe that there are different traditions in implementing mathematical modelling in classrooms. They distinguish between the implementation of modelling in everyday lessons, which are limited by strong constraints (lack of training of teachers for modelling, teaching to the test, etc.). That is the reason why, according to the authors, classroom observations regularly reveal only a low proportion of modelling. On the other hand, there is a long tradition of implementing the modelling projects when, during the modelling days or weeks, students work on one complex problem over a longer period of time. The modelling problems often come from research or industry and are slightly modified for its implementation. The participating students can choose between the several modelling problems offered. Kaiser and Schwarz (2006) explain some examples of these modelling projects (e.g., Pricing of Air Berlin, Pricing of an internet café or Risk management). They do not formulate design principles as such, but it makes clear from their description of the general aim is to promote a modelling-based understanding of mathematics and to foster students modelling competencies for carrying out modelling processes at school. At the end of

the seminar experience, the authors analyse the feasibility of implementing complex modelling problem with students, the modelling process followed and the changes in students' and teachers' beliefs and motivation.

3.2. Case study 2: Designing models development sequences in the MMP

The papers analysed in this second case study (Ärlebäck & Doerr, 2015, 2018) address the research question of how the model development sequence can support the development of students' interpretations and reasoning about negative rates of change across different contexts of physical phenomena. The whole modelling activity was designed accordingly to the "six essential principles for MEAs" (Lesh et al., 2003). Table 3 lists these design principles and how they are addressed in the experienced modelling activity. The model development sequence was implemented in a mid-size university of engineering in the United States. It was planned as an entrance course on mathematics for future engineering students. Ärlebäck and Doerr (2018) report on the third run of the course, where 35 students participated in the 6-week course. The sequence was initiated with a modelling eliciting activity about positive, negative, and changing velocity by using motion detectors to generate position graphs of the students' bodily motion along a straight path. This work was followed by a model exploration using SimCalc MathWorlds simulating motions along straight lines. The model application activities addressed how light intensity changes concerning the distance from a light source. The light intensity task was designed to provide the students with an opportunity to apply their understandings of the average rate of change, in a new context where the decrease in the dependent variable was non-linear.

Table 3. Design principles according to the models and modelling perspective

Design principles	Settling in the case study
<i>Reality (or sense-making) principle</i>	The course presents a model development sequence about average rates of change in real-world physical phenomena about changing quantities (motion along a straight line, light intensity, discharging capacitor).
<i>Model construction principle</i>	The different physical phenomena could be modelled with different underlying mathematical models (piecewise linear, inverse square, and exponential decay). Tasks for students were to create, compare, and modify models based on functions according to the data collected through experiments.
<i>Self-evaluation principle</i>	The instructors invited the students to discuss the usefulness of their models and interpretations. Most students were good at calculating average rates of change and constructing graphical representations of the changing phenomena and their associated rate of change, but there appeared some difficulties in their interpretation in terms of the physical phenomena.
<i>Model documentation principle</i>	Students in pairs were asked to deliver a written report at the end of the model application activity on light intensity. They were asked to describe parts of their modelling process: values of the light intensity and models considered, graphical and numerical interpretation of the average of rates of change, etc.
<i>Model generalization principle</i>	The model development sequence was designed to

	help students develop a general way of thinking about changing quantities in real-world contexts, instead of a solution for a specific context. By using physical phenomena (motion, light intensity, discharging capacitor), students drew on previously developed models to build more advanced models.
<i>Simple Prototype Principle</i>	The different activities helped students to create and interpret models of changing phenomena, with the same underlying structure: interpreting rates of change. The results showed how such comparisons become more complex when the rates are negative and increasing, in the case of the light intensity and the discharging capacitor contexts.

The models and modelling perspective, unlike most approaches to modelling, explicitly defines some design principles. Some of them refer to basic epistemological assumptions, such as that modelling activities are designed as a continuous sequence of models that are created, interpreted and developed across contexts. Consequently, the sequence of modelling eliciting, exploration and application activities take longer to be coherently developed. In the papers considered, the implementation in a bridging course from secondary to university is reported. This a favourable institutional context as one does not have a strict syllabus to accomplish. However, the openness of the tasks and the students' responsibilities in the activities, might be challenging to fit into ordinary teaching, which is a shared trait with our next case study.

3.3. Case study 3: Designing Study and Research Paths in the ATD

In the third case study with the ATD, Lucas (2015) presents an investigation about a possible *raison d'être* of elementary differential calculus (including differential and integral calculus) in the transition from secondary to university level. This study gives a construction of a reference epistemological model about how to interpret functional modelling¹ and how to delimit levels of functional modelling, which can address and make sense to the teaching and learning of differential calculus (Fonseca, Gascón & Lucas, 2014; Lucas, Fonseca, Gascón & Schneider, 2020). A critical result of their research was to use this reference epistemological model describing functional modelling to design teaching proposals in multidisciplinary contexts. These designs nurtured participants to go through different steps of levels of functional modelling, to connect the domains of differential calculus, algebra, and linear programming.

Based on this theory construct, Fonseca et al. (2014) design *study and research paths* (SRPs) for teaching differential calculus at secondary and tertiary levels. They describe a SRP starting from a generating question on how to analyse the propagation of influenza in areas of South Africa and the effectiveness of medical treatments. From this question, many derived questions emerged and made modelling progress:

*Q*₁: How can we model the variation of medicine concentration in patients' blood? How does this concentration vary with the time?

*Q*₂: If we want to compare the blood concentration of the medical treatment in two patients (same quantity), do both have the same evolution with the time?

¹ "Functional modelling" is interpreted as the modelling process based on the consideration of functions as models to fit (discrete or continuous) data and to provide short-, medium- and long-term forecasts.

This SRP was implemented with first-year university students of the nuclear medicine degree in a regular course of Biomedical and Radiation Sciences I in the Instituto Politécnico do Porto (Portugal). The contents of the course were adapted, beforehand, according to the necessities of the SRP. This was a result of joint work between the lecturer of the course and the researchers who were responsible for the SRP implementation. Students had two credits (out of 15) devoted to “project-based learning work” where the SRP was implemented. Students engaged in the SRP for a total of 56 hours, 25 hours at the university and the rest as online meetings. The task design was guided by the main traits of SRP (Chevallard 2006, 2015). Table 4 summarizes the design principles upon which the SRP was designed and how they are considered in the case of the influenza epidemic.

Table 4. *Design principles in the SRP*

Design principles	Settling in the case study
<i>Designed from a reference epistemological model of the praxeologies at stake</i>	The SRP was designed from a previously delimited REM about what modelling is (under the lenses of the ATD), and about what functional modelling is and the different levels or stages of functional modelling.
<i>A generating question as the starting point of the SRP</i>	Q_0 : To stop the propagation influenza A virus subtype H1N1 in 2009, how can we analyse the effectiveness of specific treatments with patients?
<i>Arborescence of questions and answers as the SRP structure</i>	The <i>a priori</i> design of the SRP is sketched through a map with possible modelling paths to be followed. The map contains the hypotheses, derived questions from Q_0 , mathematical models built and possible answers. It integrates paths and rounds of modelling, involving praxeologies of increasing complexity. This initial design is used for the analysis during and after the SRP implementation, which enriches its initial structure.
<i>Collective production of a final answer</i>	Students work in teams of 3-4 members. Each team started with the same Q_0 , but they all followed different paths. Intermediate and final reports are asked to each team, also written and oral exposition. Group (70%) and individual (30%) evaluation ran over implementation with some criteria fixed in advanced about the modelling process followed (hypotheses, models, technology for models simulations, interpretation of the models in context, coherence on answers, etc.)
<i>Collaborative and shared study process</i>	The teacher acted as a guide, organising team work, proposing questions to help students progress, moderating the discussion and participation in class and online meetings. Students in groups organised their work inside and outside the classroom, were asked to record their work in class and online meetings, to report and share with others by means of a <i>portfolio</i> .
<i>Accessible resources and evolving learning environment</i>	The official course ran at the same time as the project-based learning work with the SRP. Some contents (functions, variation rates, differential questions, interpolation, etc.) were used in the context of the SRP. Other contents were introduced by the researcher-instructor based on necessities arising during the SRP,

e.g., use of computer tools. Students used GeoGebra/wxMaxima to simulate models, fitting models to data sets forecasting short- and long-term evolutions. They were asked to look for information outside (website, library) to be shared and discussed in class.

Although this SRP was only experienced during two academic years, the initial designs and, especially, the broad reference epistemological models about functional modelling have been the base of several other designs with other generating questions (about epidemics, social networks, etc.). Designing, implementing, analysing, redesigning are essential steps in the research methodology followed –a “didactic engineering” process (Barquero & Bosch, 2015)–, to study the initial didactic phenomena detected and to analyse the potential of the designs.

4. Discussion and conclusions

One aspect that becomes clear from the analysis of the selected case studies is that each research framework holds different reference epistemological model (REM) for mathematical modelling and that this REM influences the design of modelling tasks. To deal with our research questions, we have used two strategies. The first strategy has been to describe the REM that each framework defines to approach modelling. The second strategy consists in extracting the design principles from some selected papers developed under the umbrella of the considered research framework. We want to underline two main contributions of our analysis.

The first contribution comes from the analysis of the different REM about mathematical modelling. We observe important differences, but also some similarities. From the lens of considering intermediate and domain-specific frame levels (Kieran et al., 2015), we consider that the “modelling cycle and competencies” approach could be interpreted as a domain-specific frame developed to deal with mathematical modelling as a particular content and procedural knowledge. Efforts in this approach are made on proposing a prescriptive description of mathematical modelling, viewed as a specific process aimed for in teaching mathematical modelling, to accurately analyse the cognitive process of individuals when carrying out modelling activities (Borromeo Ferri, 2007), and to make suggestions about how to design modelling tasks to support students and teachers to progress in these activities (Maaß, 2010).

The other approaches, “models and modelling perspective” and “anthropological theory of the didactic” work with wider REMs, beyond school domain-specificity. Both approaches could be considered as intermediate frames, in the sense of Kieran et al. (2015). They develop their tools while questioning what mathematical modelling is and how it can be used to provide a functional and connected study, in school mathematics, of larger questions. This could lead us to think that, under these perspectives, the specificity of what mathematical modelling is could be lost. For ATD, the conceptualisation of modelling considers that any mathematical activity can be interpreted in terms of modelling, by including intra-mathematical modelling (modelling of mathematical systems) and by considering that mathematics (as other disciplines) provide models to progress on the study of questions. However, in each SRP, there is a concrete REM of the content at stake –e.g., elementary differential calculus in the case considered– a REM beyond the domain and content delimitation prevailing in most schools or university institutions. Thus, despite the above differences, the approaches can coexist if we are aware of the REM we are assuming by our choice of approach– and that others exist. Our choice of REM must reflect the

purpose for which we design modelling tasks: assessing students' modelling competences, creating interdisciplinary teaching across school disciplines, as a vehicle for learning other areas of mathematics or whatever our research focus might be.

The second contribution is on the impact that REMs have on the design principles for modelling tasks. We can better talk in terms of the dialectic between theoretical development and task design principles, instead of one-directional impact. In the three cases, we have selected some papers to analyse the design principles upon which modelling tasks are designed. Concerning the “modelling cycle and competencies”, when designing modelling tasks, a central design principle is the potential that tasks have to promote a part, or all, of the modelling cycles or competencies. In the “models and modelling perspective”, proponents have been remarkably consistent in defining the “six essential principles for MEAs”. They show an evident dialectic between the development of the theoretical tools (such as, definition of model eliciting, exploration and application activities, or the model sequence development) and the principles that support the design of teaching sequences. In the case of the ATD, within the consideration of the “study and research paths”, several design principles appear (generating questions, arborescence of questions and answers, models or praxeologies of increasing complexity, etc.), closely related to the reference epistemological models used to analyse mathematics and, in particular, the specific modelling process at stake.

From our analyses, we see that both theoretical frameworks and design principles are dynamic entities, which develop under experimentation and through questioning the prevailing ideas on teaching and learning of mathematical modelling at all levels of the educational system, whether modelling should be understood as a means to learn content knowledge or as content knowledge itself. Furthermore, we need to be careful when approaching case studies from the lenses of other frameworks than those where they were produced, because of the dependence between reference epistemological models and design principles. Moreover, in line with Kieran et al. (2015), despite the recent growth number of design studies within mathematics education, the specificity of the principles that inform task design in a precise way remains both underdeveloped and, even when somewhat developed, underreported. More work remains to be done that aims at looking jointly at frameworks, task design principles, and local ecologies for implementing mathematical modelling.

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Impact of theoretical perspectives on the design of mathematical modelling tasks

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The focus of this paper is to discuss how the adoption of a particular theoretical framework affects task design in the research on modelling and applications. To analyse and compare frameworks within this research, we draw on tools from the anthropological theory of the didactic and the notion of reference epistemological model. We start by outlining the existence of different reference epistemological models about mathematical modelling to analyse the impact they have on the principles guiding the design of modelling practices. We select three theoretical perspectives: the modelling cycle and competencies approach, the models and modelling perspective and the anthropological theory of the didactic, to investigate relations between the reference epistemological models each approach builds and the corresponding design principles for modelling. We restrict our analysis to three case studies as representatives of the three perspectives. We are conscious that the three perspectives do not cover the research on “modelling and applications”, nor the three studies represent all design principles embedded in each framework. Still, the chosen approaches display the variety of meanings of the term and why we need to be cautious with not assuming general consensus. Each case study is examined through the selection of some research papers, following two main analytical strategies. The first strategy is to describe the reference epistemological model that each framework defines to approach mathematical modelling. The second strategy is to extract the design principles from the selected papers developed under the particular framework. For each case study, we inform about the modelling activity proposed to underline and exemplify the design principles and finally to inform about the local “ecologies” created for implementing modelling. We conclude with some contributions from the comparative analysis across cases. The first contribution comes from the analysis of the different reference epistemological models about mathematical modelling. From the lens of considering intermediate and domain-specific frame levels, we can detect important differences, but also some meeting points amongst approaches. The second contribution is on the impact that adopting a particular theoretical perspective has on the design principles for modelling tasks or the dialectical relation between the theoretical assumptions adopted and the task design principles used. We see how theoretical frameworks and design principles are dynamic entities, which develop for designing and implementing modelling in school contexts and through questioning the prevailing ideas on teaching and learning mathematical modelling.