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Mathematical models and meanings by school and university students in a modelling task

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Abstract

This study involves two classes from different educational levels, namely 9th grade and university. Students in both contexts were given a modelling task that required the development of a hand biometrics recognition system, during which they performed experimentation and simulation. As aims of the study, we look for distinctions and commonalities between the models developed in the two classes and seek to know how simulation and experimentation influence students' production of meaning. The theoretical framework comprises the relationship between the modelling process and the prototyping process and adopts Peirce's pragmatic perspective on meaning. The research is of a qualitative nature, assuming the characteristics of a case study. The results reveal many commonalities between the modelling in the two contexts. Moreover, experimentation and simulation were relevant elements for the production of meaning by the students, which is endorsed by a pragmatic perspective on meaning.

Keywords. Mathematical modelling; meaning; prototype; simulation; semiotics.

Modelos matemáticos y significados de alumnos y de universitarios en una tarea de modelización

Resumen

Este estudio involucra dos clases de distintos niveles educativos, uno en secundaria y otro en la universidad. A los estudiantes en ambos contextos se les asignó una tarea de modelización que requería desarrollar un sistema de reconocimiento biométrico de la mano, durante el cual hubo experimentación y simulación. Como objetivos de nuestra investigación, buscamos diferencias y puntos en común entre los modelos desarrollados en estas clases y tratamos de saber cómo experimentación y simulación influyen en la producción de significado de los estudiantes. El marco teórico contempla la relación entre el proceso de modelización y el proceso de creación de prototipos y adopta la perspectiva pragmática de Peirce sobre el significado. La investigación es de corte cualitativo con características de un estudio de caso. Los resultados revelan muchos puntos en común entre la modelización en los dos contextos de experimentación. Además, experimentación y simulación fueron elementos relevantes para la producción de significado por parte de los estudiantes, lo que está respaldado por una visión pragmática sobre el significado.

Palabras clave. Modelización matemática; significado; prototipo; simulación; semiótica.

1. Introduction

Despite clear evidence that the integration of mathematical modelling in mathematics classes can take on different configurations and aims, as well as reflecting different educational perspectives, there is agreement on the fact that modelling is cognitively demanding for students (Blum, 2015). This cognitive demand is intrinsic to the modelling process, not only because the construction of a mathematical model of a real world situation requires mathematization, but also because several other cognitive processes take place, all of them linked to possible obstacles. While the cognitive demand is undeniable, there is awareness that modelling tasks offer opportunities for learners to construct meaning, both mathematical and extra-mathematical (Silva & Almeida, 2015).

Mathematical modelling tasks can be beneficial for different students. This assertion has been repeatedly confirmed by several researchers. We refer here to the example presented by Borromeo Ferri (2018) based on the idea of one modelling problem for all. Her research showed what happens when the same problem (organizing a class party) was solved by different elementary school students, from the first to the fourth grade. The self-differentiating character of the problem revealed that the mathematical model and the mathematical working were very diverse from grade to grade, as well as the mathematical results and the final results, even though they were validated on the basis of the student's assumptions. The results showed that despite the mathematical working diverged between the various groups, the construction of meanings was essential in all of them, especially in the interpretation of the situation and in the formulation of meaningful assumptions.

Other examples have been presented by researchers who propose the use of modelling situations in which an engineering problem is posed. In many cases, students have to create mathematical models to respond to a client's request. These are typically problems where students are challenged to create a meaningful product, which includes building a model of its essential structure. Such types of modelling problems have been investigated at various levels of education, including primary and middle school (English & Mousoulides, 2011). An important aspect of student's work on such problems is the assessment of the appropriateness of the models produced in satisfying the requirements of the client. This leads to the importance of students carrying out experimental or practical work and being able to simulate the functioning of the system in a realistic way.

In this article, we examine the work on a modelling task carried out by different groups of students and from different educational levels. In particular, we will focus on particular aspects of their work, such as experimentation and simulation, to understand how they may impact on the meaning production in the modelling process. We will address two research questions: 1) What similarities and distinctions can be identified between the models produced by 9th graders and those produced by university students in a modelling task? 2) What is the influence that experimentation and simulation have on the construction of meaning in the modelling process?

2. Theoretical framework

2.1. Mathematical modelling from a cognitive perspective

The mathematical modelling process is typically described as a cycle composed of several steps, as Blum and Leiß (2007) suggest: (1) Understanding the real problem; (2) Simplifying the original situation; (3) Mathematizing; (4) Working in the mathematical domain; (5) Interpreting the results obtained; (6) Validating; (7) Presenting the results.

The general representation of the modelling process has become widespread in the research on teaching and learning mathematical modelling, regardless of the variants that have emerged to emphasize certain features of the steps involved. It has been considered as particularly helpful in guiding cognitive analyses of the students' thinking and meaning production in carrying out a modelling task (Blum, 2015; Borromeo Ferri, 2018).

One of the current perspectives embraced by researchers in the study of modelling in mathematics education is the cognitive perspective. In some studies, this perspective concerns the analysis of the students' processes while undertaking a modelling task, usually framed by the specific cognitive processes taking place in the modelling cycle; this also includes identifying difficulties, barriers, and meanings or ways of understanding the situation, the problem, the model, and the results.

For some authors, a cognitive view concerns understanding the micro-processes in students' modelling and deeply analysing their thinking processes, also referred to as individual modelling routes (Borromeo Ferri, 2018). Others adopt a more longitudinal approach to the analysis of the cognitive activity of students engaged in a modelling task and aimed at tracing the path of the meaning production linked to the various stages of the modelling cycle (Silva & Almeida, 2015). This cognitive perspective involves analytical tools that include the mathematical modelling cycle as an underlying structure along with tools supported by diverse theoretical structures, such as a semiotic theory.

2.2. Models in engineering design

Mathematics is as indispensable for solving real world problems as it is for creating and producing all kinds of industrial products that are part of our daily lives. Many objects, machines or devices are results of a process called industrial design, engineering design or, simply, prototyping. A key idea in the engineering design process is that engineers need to model, in some way, the processes and devices they will create (Dym, 2004). The models, especially those mathematical, are essential in predicting what the behaviour of a system or process will be before the actual engineering product is made.

Briefly put, prototyping is the process that leads to the simplified realization of a concept or idea in an operational product that reproduces essential aspects of the behaviour of the final product. The cyclical character of the prototyping process is generally represented through the "prototyping cycle" (Figure 1). It may be seen that mathematical modelling and prototyping are very similar, in many respects. Isa and Liem (2014) explain that mathematical modelling and prototyping are closely associated. While the first aims to generate a mathematical model of something real, the later aims to study and test how a new product will be used and how it will work in its manufactured state. In all engineering domains, prototypes include mathematical models and computational models. Both models and prototypes share similarities with some targeted system. In addition, both are indispensable representation tools, allowing expressing ideas and concepts as accurately and effectively as possible.

Two important notions in the engineering design are those of experimentation and simulation. According to Birta and Arbez (2007), "the modelling activity creates an object (i.e., a model) that is subsequently used as a vehicle for experimentation. This experimentation with the model is the simulation activity" (p. 4).

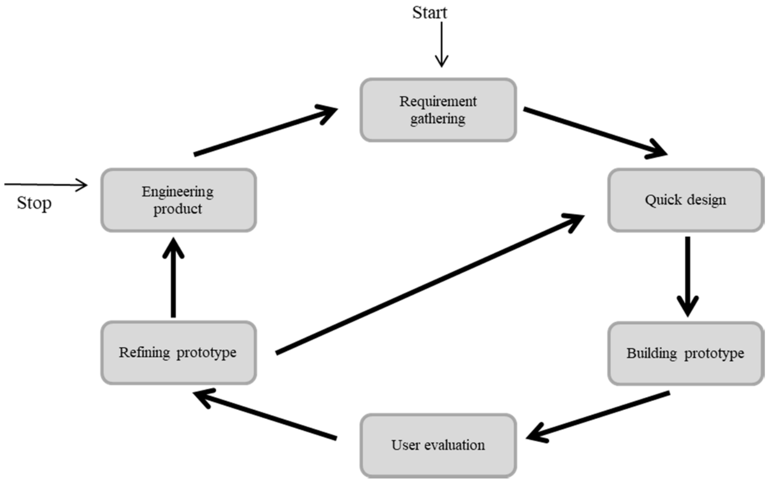


Figure 1. The prototyping cycle (adapted from Laudon & Laudon, 2012)

One of the reasons for considering modelling and simulation in a project design is the intention to carry out prototyping and concept evaluation. The ideas of modelling and simulation in a project are related to the so-called experimental frame of the project. Because the system is intended to have an intended behaviour, there is the need to evaluate that behaviour by means of testing a prototype. The prototyping and concept evaluation can be carried out by means of the construction and validation of models.

In this view, an appropriate model (particularly, a mathematical model) for the system under investigation is a representation or abstraction of the system. The experimental work implies developing a mechanism to ensure an appropriate compatibility among the system, the model, and the goals to be achieved (Birta & Arbez, 2007).

Svarovsky and Shaffer (2007) point out the relevance of having the students observing and manipulating real objects and materials –the experimental or practical work on the project– when solving a design challenge. They refer to rituals within the design process: brainstorming, conducting experiments, sharing design ideas with peers, building and testing prototypes, and optimizing a solution through redesign. In fact, the integration of experimental work in the development of models has also been addressed in our previous studies where such work includes conceiving procedures, which in turn lead to making simulations for seeing and interpreting how systems will work in the real world (Carreira & Baioa, 2017, 2018; Baioa & Carreira, 2019).

2.2. Meaning from a semiotic perspective

The Peircean semiotics approach to meaning production in modelling tasks is based on Peirce's pragmatism. Charles Sanders Peirce (1839-1914) constructs his theory of meaning based on pragmatism. He conceives pragmatism as a method of reconstructing or explaining the meanings of any concept, doctrine, proposition, word or other sign. In this method, the procedure adopted by Peirce to construct or explain meanings consists of establishing a set of conditions for a given situation in which a defined operation would produce a defined result. That is, to determine what a concept means or "to make clear the meaning of an idea, we must try to interpret each notion by tracing its practical consequences" (Peirce, 1972, p. 21). For example, to say that an object is hard, we should try to scratch it using different substances and, by doing that, arrive at the result: the object cannot be scratched by most of the substances applied to it. In this way, the concept of 'hard' would get a precise pragmatic meaning: 'not possible to be scratched'.

In the field, a pragmatic point of view on the meaning of mathematical objects is discussed by Wilhelmi, Godino and Lacasta (2007) who state that "our meaning begins by being pragmatic, relative to the context… the meaning of a mathematical object is inseparable from the pertinent systems of practices and contexts of use" (p. 79).

Silva and Almeida (2015) examined different studies by Peirce and by several authors who worked on his theory and concluded that the pertinent systems of practices and contexts of use are, in fact, essential aspects considered by Peirce in meaning production. Particularly, based on Peirce's theory, the meaning production in mathematics education can be associated to some relevant aspects. (1) The *familiarity* the person has with the ideas or concepts she/he is using. That is, if the person has already a broad understanding of the concepts or objects, the construction of meaning in a new situation is favoured. (2) The *intention* of the person in signifying a new idea or concept, in articulating it with the context in which it is used. (3) The identification by a person of a *future consequence* of using an idea, a concept or an object. Thus, a certain anticipation of the importance of an idea, concept or object in a given situation, can favour the production of meaning by the person in that situation. (4) The person's *collateral experience* with the idea or concept he/she must deal with in a given situation. That is, if the person has already had other experiences of using this concept, meaning production in a new context of use is favoured.

3. Empirical data and method

3.1 The two research settings and the modelling task

The present study involves groups of students from two educational contexts working on the *Hand Biometry* task (Figure 2). One of the contexts is a class of twenty 9th grade Portuguese students. The other context is a class of eighteen Brazilian university students who were attending the last year of their mathematics degree. On the two contexts the students were organized in groups to work on the task and at the end each group had to deliver a written report. Two of the authors of this article were the teachers of the classes. In each class, the task was introduced with some discussion of general ideas about hand biometry. The modelling task involved using a sample of real images of students' hands for data collection and performing simulations based on a model that would have to be created for a hand recognition system.

Hand recognition based on biometrics uses hand geometry and a set of measurements that can be extracted, considering some points of the hand (for example, the five fingertips and the four valleys between them, as the image illustrates). Your assignment is to create a recognition system for authorizing access, using the hand geometry (the system may be used, for example, to access the cafeteria or some specific room). For this, you have at your disposal a set of real-seized images of your palms. The recognition system must:

- *accept an individual that is contained in the database;*
- *reject an individual that is not contained in the database.*

Figure 2. The Hand Biometry task presented to the students in the two contexts

The development of a recognition system based on hand biometrics is a real problem in the field of electronic and computer engineering. Experts in this area (see Varchol & Levický, 2007) say that the typical architecture of recognition systems involves three phases: enrolment, pre-processing and verification. In the enrolment phase, images of the hand (templates) are captured from a sample of subjects to make a database. In the pre-processing phase, the chosen features are extracted from the images and a comparison algorithmic process is created. The comparison is based on a distance calculation model, which implies establishing a threshold value for acceptance or rejection. In the verification phase, the algorithmic model is tested to determine if a given hand image is the hand of a user. Finally, the performance of the model is assessed and possible reformulations or improvements are proposed.

3.2 Data collection and analysis

In this study the empirical data were collected in similar ways in each of the two classes. The dialogues and discussions in each group were audio recorded. Written reports and other worksheets produced by students were collected and field notes from participant observation, as well as photos of the students' work, were taken by the teachers. We selected two groups of students in each class whose approaches were distinct, particularly in the mathematical models they produced. Our analysis progressed with describing, documenting, and interpreting the data in light of the theoretical concepts that frame the study, namely: i) experimentation and simulation in the mathematical modelling process and the prototyping process; ii) the pragmatic perspective on the meaning production.

We carried out a qualitative research study (Sharma, 2013) with an interpretive nature, focusing on the students' cognitive processes while solving a modelling task in the classroom. The research study embraces a case study design, in the sense that it draws on empirical data that are collected in order to explain, explore or describe a real and complex issue (Yin, 2010). It intends to produce an understanding of a limited number of events and of their relationships through detailed and contextual analysis.

4. Results

4.1. An overview of the 9th grade students' modelling processes

A set of materials was distributed to each group in the 9th grade class: full-size photocopies of their hands, worksheets for creating a database and a guide for preparing the report. The students started by creating a biometric database for the members of their group. They decided which features they would select to create a prototype for the hand recognition system. The choices of the extracted features were very diverse, including the lengths of fingers and widths of the phalanges, and the palm area.

After some discussion, all groups realised that a perfect match between any incoming hand and the biometric data stored in the database would be highly unlikely. They put aside the notion of perfect match and decided to assume a reasonable match; this led them to establish a threshold value, which some called the "error margin", in the comparison process. Then they created a mathematical model (algorithmic model) to carry out the comparison. For the validation of their prototypes, images of unidentified hands were randomly distributed by the groups. By applying the developed algorithm to an unknown hand, the groups simulated their recognition system and assessed their models.

4.2. Types of models by 9th grade students

In all groups the idea of comparison was central. Students understood they would need to create some kind of comparison mathematical model to check the match between an unknown hand and the stored data. Mostly, the idea of acceptable error became clear.

Group 1, composed of 4 students, produced a model which we call of *exhaustive direct comparison*. The students considered two types of measures (lengths and areas) and acceptable errors (0.1 and 0.5 respectively). The linear measurements were stored in the first 19 columns of a table and the palm areas were recorded in the 20th column. So, the table had a format of 4 rows (users) by 20 columns (features). The system would work by comparing the set of features of an unknown hand with the corresponding set of features of every enrolled hand. This was made by calculating the distance, using the absolute value of the difference. The criterion for accepting an unknown hand was that for some recorded hand all the distances obtained should be inferior to the allowed errors.

This model of *exhaustive direct comparison* may be described in mathematical terms as follows. Consider the table entry x_{ij} as the value referring to the user *i* and to the hand feature j; consider y_j as the value referring to the hand feature j of an unknown hand. The unknown is authorized when there is a user *i* in the database such that for all hand features j, the conditions are true: $\left|x_{ij} - y_j\right| < 0.1$ (for linear measurements) and $\left|x_{ij} - y_j\right| < 0.5$ (for area measurements). Likewise, the unknown is not authorized when for every user *i* there is at least one hand feature j such that $\left|x_{ij} - y_j\right| \geq 0.1$ (for linear measurements) or $\left|x_{ij} - y_j\right| \geq 0.5$ (for area measurements).

Group 2, composed of 5 elements, decided to collect the lengths of each finger and the widths of the phalanges, in a total of 19 features. The students then thought about creating other parameters to improve the differentiation of the hands and to avoid the exhaustive comparison. For each finger, they calculated the sum of the extracted features (lengths and widths) and named it as the *total finger*. Their model worked with the *total fingers*, thus generating a table of parameters composed of 5 rows (users) by 5 columns (*total fingers*). For the comparison algorithm, the first step was to calculate the *total fingers* of the incoming hand. Then, for each user in de database, they calculated the absolute value of the difference between each *total finger* and the corresponding one of the unknown hand. Finally the five distances were added up. The unknown hand would be accepted if the sum of the distances were less than 1, which implies that each distance would also be less than 1. For the rejection criterion, the students first considered the denial of the previous rule but later observed that having a sum greater than or equal to 1, would not say enough about the numbers added (all of them could be less than 1, such as 0.5 + 0.4 + 0.3 + 0.2 + 0.1). So, in the case of a sum greater than or equal to 1, each of the added distances had to be checked individually. There would be rejection if at least one of the distances was greater than or equal to 1, otherwise there would be acceptance.

This *condensed comparison model* may be mathematically formalized in the following way. Consider the table entry X_{ij} as referring to the user *i* and to the *total finger j*; consider Y_j as the *total finger j* of an unknown hand. The unknown is authorized when there is a user *i* in the database such that $\sum_{j=1}^{5}\left|X_{ij}-Y_j\right|<1$. In case, there is no user in the previous case, then the unknown is authorized when there is a user i such that $\sum_{j=1}^{5}\left|X_{ij}-Y_j\right|\geq 1$ and $\left|X_{ij}-Y_j\right|<1$ for all the *total fingers* (for instance, 0.5 + 0.4 + 0.3 + 0.2 + 0.1). Finally, the unknown is not authorized when the condition $\sum_{j=1}^{5}\left|X_{ij}-Y_j\right|\geq 1$ is true for all users, and for each user there is at least one *total finger j* such that $\left|X_{ij}-Y_j\right|\geq 1$.

4.3. Meaning production by 9th grade students

In the case of Group 1, the students were concerned with the need to individualize each hand. They talked about the importance of a detailed description of the hand, which explains the large number of features they selected. In fact, they collected the lengths of the 5 fingers, the lengths of the phalanges and also the area of the circle circumscribing the palm, in a total of 20 measurements (Figure 3a-b).

Then the students considered ways to perform comparisons involving the data recorded in the table. Initially, they seemed to hesitate on the data to use for the comparisons, as the dialogue indicates:

Teacher: Are you going to compare all of those [features] in your mathematical model?

S3_1: Maybe.

S2_1: Well, this is what we have here! [referring to the table]

Teacher: Ok.

By examining the values obtained for each hand feature, they found that some values appeared more than once and decided that the best approach would be an exhaustive comparison for every user, by calculating the distance, for each feature, and checking whether or not it exceeded the acceptable errors (one for the lengths and one for the area).

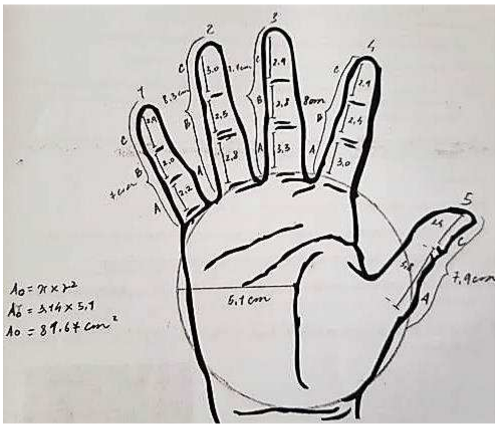


Figure 3a. Image from Group 1 report

Figure 3b. Image of the photocopy of a hand showing a sketch for data collection

The group tested their algorithm with an unknown hand and concluded that it was an unauthorized person, right after the first comparisons showing higher distances than accepted, as their dialogue shows. Thus, the meaning associated with verification was finding out whether each of the calculated distances was within the accepted tolerance.

S1_1: Okay, you have to check them all [the features of the unknown hand].

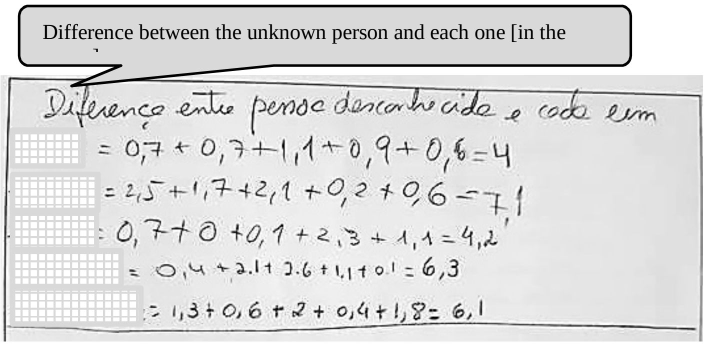
S1_1: This is false. It is not in our group.

S2_1: We have a close number here, but it is not equal.

S3_1: No, not within the error margin. Definitely not.

The students of Group 2 extracted the lengths of the fingers and the widths of the lines that separate the phalanges. Later, they decided to use, for each finger, the addition of all the values collected, as an attempt to reduce the number of comparisons. In the students' report they named those totals as the *total fingers*.

The model required the calculation of the distances between the *total fingers* of an unknown hand and the respective *total fingers* of each hand in the database. Then, the sum of the distances would describe the difference between the unknown hand and a hand from the database, as shown in the group's report where the calculations refer to the actual test performed (Figure 4).



Difference between the unknown person and each one [in the

Diferença entre pessoa desconhecida e cada um

[redacted] = 0,7 + 0,7 + 1,1 + 0,9 + 0,6 = 4

[redacted] = 2,5 + 1,7 + 2,1 + 0,2 + 0,6 = 7,1

[redacted] = 0,7 + 0 + 0,1 + 2,3 + 1,1 = 4,2

[redacted] = 0,4 + 2.1 + 2.6 + 1,1 + 0.1 = 6,3

[redacted] = 1,3 + 0,6 + 2 + 0,4 + 1,8 = 6,1

Figure 4. Excerpt of Group 2 report (on the left, the names of the students were shielded)

Finally, they came up with conditions for the sum of the distances, for the case it was less than 1, and for the case it was greater than or equal to 1. The following dialogue reveals how the students concluded that their test determined a rejection.

Teacher: So the [unknown] hand was it rejected or accepted?

S5_2: It was rejected.

Teacher: Why?

S5_2: Because, in our way of comparing, any of the sums exceeds our tolerance and there are also values in each of the sums that exceed the tolerance.

The model of this group attempted to condense several hand features, through adding, in order to obtain a description of each finger (the *total finger*). At the same time, the model introduced a measure of the distance between two hands, using the *total fingers*. The sum of the distances between the *total fingers* represented the measure of the distance between two hands. However, the measure of the distance between hands was not enough to ensure the established tolerance and the model had to include additional conditions that meant doing supplementary comparisons.

4.4. An overview of university students' modelling processes

The aim of the task was discussed at the beginning of the class and the students agreed on the idea of creating some type of algorithm (model) that would allow determining whether a person belongs to a particular group or not. The decision was that each group would develop a security system so that the members would have authorized access through that system and the other students in the class would be considered intruders.

From that initial point of shared understanding of the problem, the groups started their experiments with regard to the data collection process and to the process of building an algorithm. Having achieved a prototype of the recognition system, each group performed tests to check their validity. We will discuss the models developed by two groups.

4.5. Types of models by university students

Group A, composed of 4 students (X_1, X_2, X_3, X_4), initially took 15 measurements from the hands of the 4 members (Figure 5a). Then the students started to explore ways to construct a model and after some time they realised that it was difficult to deal with too many features. After some experimentation, using the initial features, the group chose to use 9 of them (Figure 5b).

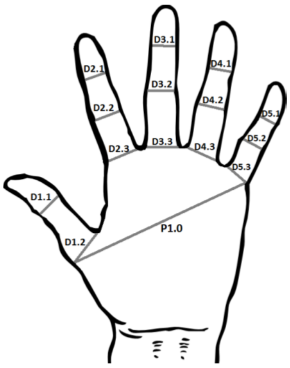
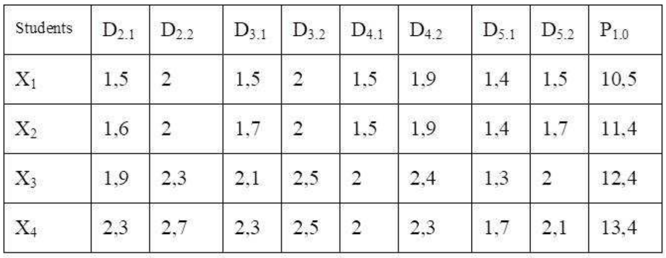


Figure 5a. The hand measures



Students	$D_{2.1}$	$D_{2.2}$	$D_{3.1}$	$D_{3.2}$	$D_{4.1}$	$D_{4.2}$	$D_{5.1}$	$D_{5.2}$	$P_{1.0}$
X_1	1,5	2	1,5	2	1,5	1,9	1,4	1,5	10,5
X_2	1,6	2	1,7	2	1,5	1,9	1,4	1,7	11,4
X_3	1,9	2,3	2,1	2,5	2	2,4	1,3	2	12,4
X_4	2,3	2,7	2,3	2,5	2	2,3	1,7	2,1	13,4

Figure 5b. The features of the students' hands

Using these nine features they defined a new parameter using matrices and determinants. The value of the parameter was calculated for each member in the group, as follows:

$$\alpha_i = \left| det \begin{bmatrix} D_i 2.1 & D_i 3.1 \\ D_i 2.2 & D_i 3.2 \end{bmatrix} \right| + \left| det \begin{bmatrix} D_i 4.1 & D_i 5.1 \\ D_i 4.2 & D_i 5.2 \end{bmatrix} \right|$$

This yielded: for X_1, $\alpha_1 = 0.41$; for X_2, $\alpha_2 = 0.31$; for X_3, $\alpha_3 = 0.96$; and for X_4, $\alpha_4 = 0.75$.

The students then used the measures of the palms, recorded in the last column of the table (Figure 5b) and noticed that the values were less than 12 for the girls and greater than 12 for the boys. Those data were included in the model by creating another parameter for each member of the group, defined as $\alpha \times z$, where z is the value recorded in the column labelled as $P_{1.0}$. They obtained: for X_1, $\alpha_1 \times z_1 = 4.305$; for X_2, $\alpha_2 \times z_2 = 3.534$; for X_3, $\alpha_3 \times z_3 = 11.904$; and for X_4, $\alpha_4 \times z_4 = 10.5$.

The algorithm presented by the students as the model for hand recognition was expressed as follows.

An individual is authorised, according to the rule:

If $z < 12$, then $\alpha \times z \in [3.5, 4.5]$

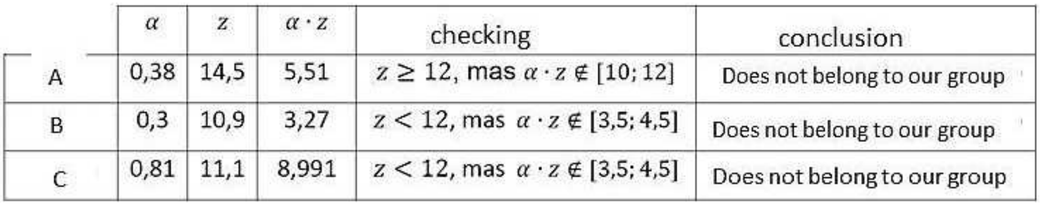
If $z \geq 12$, then $\alpha \times z \in [10, 12]$

They checked the access for each of the 4 students of the group, using the security system. In Figure 6, the process of verification performed by the students is presented.

STUDENTS	α	z	α.z	CHECKING	CONCLUSION
X_1	0,41	10,5	4,305	z<12 and α.z €[3,5;4,5]	Belongs to our group
X_2	0,31	11,4	3,534	z<12 and α.z €[3,5;4,5]	Belongs to our group
X_3	0,96	12,4	11,904	z≥12 and α.z €[10;12]	Belongs to our group
X_4	0,75	13,4	10,5	z≥12 and α.z €[10;12]	Belongs to our group

Figure 6. The validation of the algorithm

The students decided to check if their prototype would detect possible intruders from outside the group. For this, the 9 measurements of other 3 students in the class who were not members of the group were obtained. The process of verification, as shown in Figure 7, confirmed that the three outsiders were in fact intruders and therefore access would be denied. In Figures 6 and 7 we show the translated reproduction of the students' report.



	α	z	$\alpha \cdot z$	checking	conclusion
A	0,38	14,5	5,51	$z \geq 12$, mas $\alpha \cdot z \notin [10; 12]$	Does not belong to our group
B	0,3	10,9	3,27	$z < 12$, mas $\alpha \cdot z \notin [3,5; 4,5]$	Does not belong to our group
C	0,81	11,1	8,991	$z < 12$, mas $\alpha \cdot z \notin [3,5; 4,5]$	Does not belong to our group

Figure 7. The verification of the outsiders

Group B, also composed of 4 students, decided to use 8 hand measurements, which are shown in Figure 8a. This group used the Tracker software to collect the features of the hands (Figure 8b). Additionally, they created a new parameter that was defined as the sum of the 8 measurements:

$$A9 = A1 + A2 + A3 + A4 + A5 + A6 + A7 + A8$$

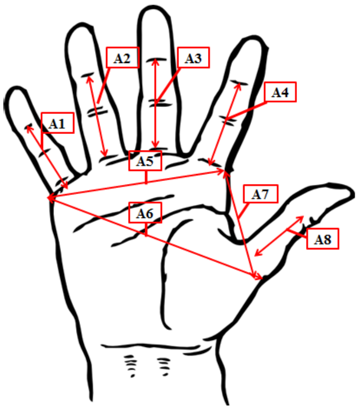


Figure 8a. The hand measures

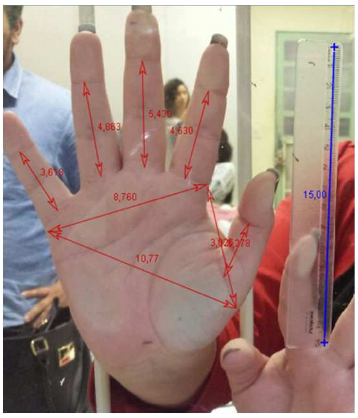


Figure 8b. The use of the software Tracker

The group turned to the construction of the security system. The system included a matrix I, composed of the data obtained for each member of the group, a matrix C, intended as a coding matrix and created at random, a matrix P defined as the product of I by C, and finally the determinant of the matrix P:

$$I\colon \begin{bmatrix} A1 & A4 & A7 \\ A2 & A5 & A8 \\ A3 & A6 & A9 \end{bmatrix} \qquad C\colon \begin{bmatrix} 1 & 5 & 21 \\ 2 & 8 & 34 \\ 3 & 13 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} A1 & A4 & A7 \\ A2 & A5 & A8 \\ A3 & A6 & A9 \end{bmatrix} . \begin{bmatrix} 1 & 5 & 21 \\ 2 & 8 & 34 \\ 3 & 13 & 1 \end{bmatrix}$$

Using the determinant and the rounding up and down by 0.05% of its value, an encoded database for the group members was created. Figure 9 shows the clients database.

Clients database - Codes			
Exact		Lower	Upper
Client 1	62304,2658	62273,11367	62335,41793
Client 2	45914,688	45891,73066	45937,64534
Client 3	39072,24	39052,70388	39091,77612
Client 4	67873,464	67839,52727	67907,40073

Figure 9. The encoded database of the clients (translation of students' report)

The algorithm for determining whether a person belonged to the group was defined: having I as the matrix of the features of the hands to be evaluated by the system, an individual will be accepted if the determinant of P belongs to at least one of the intervals in the system database; otherwise, the unknown person will be considered an intruder.

4.6. Meaning production by university students

Both groups of students had an initial concern about the number of hand measures they should use to build a robust algorithm that prevented false acceptances and false rejections. Group A, which initially took 15 hand measurements, when asked by the teacher about possible difficulties in using all of them in their algorithm, answered:

> *As we initially did not know which algorithm we were going to build, we decided to take several measurements and later choose which would be useful for our model. So, we did not decide at the outset which of the measures would be used. We left that decision to be taken in the next steps, when we would think about possible models that could be produced* (audio recording transcription).

The students felt the need to work with the measures in an experimental way to decide on the features they should keep, as shown in Figure 5. Therefore, compatibility between the data, the model, and the goal was one of the meanings assigned to the construction of their model. Later, the results of their experiments with the selected features led them to define new parameters, α, z, and $\alpha \cdot z$, as a way to increase the efficiency of the algorithm in detecting small differences among different persons' hands. The several operations performed in the algorithm indicate that the students were aware of the importance of processing the raw data and, as such, the meaning of the algorithm is associated with the aim of distinguishing between hands that may look similar.

Group B assumed that taking measurements with a ruler could lead to errors that might undermine the precision of the model. In their report, the students explained their decision to use software to capture the hand measurements. In this case, the students' familiarity with the software has largely contributed to the meaning production in the process of obtaining the algorithm. The experimental process of this group for creating a database and developing an algorithm led to operations with matrices and determinants, as well as to the use of technological tools such as smartphones, software and computers. The choice of what they called the 'error margin' aimed to ensure the efficiency of the algorithm and was the result of a simulation process in which the use of tools was relevant for the work of testing and assessing.

The university students used mathematical and technological tools that are in line with their academic level and previous mathematical knowledge. That use indicates some familiarity and collateral experience with the specific mathematical concepts and conditions they integrated in their mathematical models. The meaning that was produced in obtaining the models is also related to that familiarity and collateral experience.

5. Discussion of results

5.1. The models and meanings of students from different educational levels

Our two sets of data reveal several common features between the modelling processes of 9th grade students and of university students. Both understood the aim of generating a prototype for an authentication system based on hand biometrics. It is interesting to note that in both cases the students engaged in the three steps assumed in such a system development: enrolment, pre-processing, and verification.

Regarding the enrolment step, in both cases a concern with collecting a wide range of hand measurements was clear. There were evident similarities in the kinds of measurements in use by the two categories of students. Also, in both cases, the features collected were defined as to ensure the system's ability to differentiate hands from different people. The students perceived the risk of two people having hands that closely resemble in their measures. Thus, the intention to introduce details in their database was obvious. The purpose of creating a secure system, that is, a system that would not generate false authorizations was also shared, which shows that in both educational levels the students developed a meaning for the security of the prototype they were building.

The search for a security system that would allow an acceptable tolerance was also a dominant focus in the two classes. The students quickly understood that the system should compare the data of an unknown hand with the data of the hands registered in their database to then generate a result. Thus, the meaning of a systematic comparison process was evident in the models of all the students. In either case students understood that they would need acceptance/rejection criteria and that such criteria would have to be based on the manipulation of the quantitative information that was stored in their databases.

For the 9th grade students, building a mathematical model meant finding rules that would allow them to accept or reject a hand. Those rules were formulated according to their mathematical knowledge. They were based on the description of a systematic procedure using the distance between values. The two groups differed mainly in the degree of pre-processing of the collected features; one of the groups defined new parameters to be used in the comparison model. This decision was apparently motivated by the search for a more shortened algorithm than the exhaustive comparison. The types of mathematical algorithms created by the students drew on elementary mathematics; however they revealed meaning through expressing a comparison procedure.

For the university students, creating additional parameters from the raw data was a distinctive process, which emerged from the more extensive experimental work they carried out with the data. Apparently the students immediately assigned to their model the meaning of data processing. Although their objective was, as for the younger students, to create a system that would allow comparisons within acceptable tolerance, they chose even more economical ways of generating parameters to characterize the hands of ‘clients’. The university students, looking for mathematical procedures and concepts to construct the model, associated the data with matrices and for the construction of the model they used the determinant. Not only did they create more elaborated algorithms but used greater mathematical sophistication. Nevertheless, the meanings attributed to the produced models, in their general principles, were similar and comparable in the two groups of students. The underlying mathematical principles of the models, in both cases, involved calculating distances and defining a threshold value or acceptable tolerance.

Finally, with regard to the stage of verification or testing the prototypes that were created, the students in both classes performed simulations using hands from one or more colleagues. In any of the situations, such simulations had a key role in the validation of the models and gave students a sense of the validity of the prototype they had imagined.

The results reveal that the processes involved in the modelling cycle as well as those in the prototyping cycle were performed by students in both educational levels. The several students developed good understanding of the real problem, which entailed the construction of a prototype of a hand recognition system. They identified the relevant elements and the data to be collected in the real world, constructed a valid conceptual model of the way the system should work, and considered constraints, risks, and security levels required in the real situation. Those were translated into mathematical ideas and procedures, under the form of algorithms for acceptance/rejection. The validation took place by means of simulating the system prototype with one or more unknown persons.

The processes carried out by students showed that the problem proposed was self-differentiating and provided them with diverse mathematical knowledge and proficiency to generate assumptions, models, and results. The complexity of the task did not inhibit students with a lower level of mathematical knowledge from proposing ideas and models that have appropriate meaning and reflect the type of thinking involved in the architecture of hand recognition systems. In general, the commonalities between the school and the university students' modelling processes were stronger than one could expect, despite the different mathematical concepts and tools that were employed by the students in each educational level. This emphasizes the idea that complex modelling problems offer to diverse students the opportunity to work, in meaningful ways, with the mathematical aspects of the modelling process and with the aspects of the reality that are crucial to the development of the model (Borromeo Ferri, 2018; Blum, 2015; Almeida, 2018).

5.2. The role of experimentation and simulation in creating a system prototype

The experimentation carried out by the students from both educational levels during the construction of their mathematical models is linked to the way they established a connection between mathematics and the real situation of hand recognition based on biometrics. In fact, the elaboration of the models was based on an experimental approach in which data were collected and mathematical and technological tools were used together. The established features for hand recognition and the structuring of the models took place within an experimental process.

The symbolic language in the production of the prototypes was shaped as the real situation and the mathematics were being associated. Even the youngest students who used informal language proved to be able to mathematize the situation through logical and algorithmic thinking. As already discussed by Carrejo and Marshal (2007), and Levy (2016), the conclusion that a model can be considered good enough is the result of experimentation and simulation carried out by the students. The validation of the models was attained by testing the prototype developed by each group, through verification tests.

In the course of the mathematical modelling task, experimentation moved ideas and thoughts into actions. Those actions were building and evaluating mathematical models, and the prototypes based on them. The action of experimenting played a role in understanding the real situation and its conditions, showing future consequences that gave meaning to the mathematical model; similarly, the simulation based on the mathematical model showed future consequences in the results generated by the model. This influence of experimentation and simulation is aligned with elements of pragmatism, as put by Peirce (1972) in his deliberations on meaning production. In student meaning production in the modelling process, we could see collateral experience with diverse tools and knowledge as well as the option of verifying future consequences of ideas and thinking.

6. Final conclusion

In this article we reported the cases of two classes – 9th grade and university – solving a modelling task that required the development of a hand biometrics recognition system. We presented the analysis and interpretation of the models produced in each of the classes. The results allow us to conclude that there are many common aspects between the types of models developed by school students and by university students. They all took similar approaches to the real problem and had in mind the development of a prototype of a real system that should work properly. Such a prototype involved the construction of a biometric database and of an algorithmic mathematical model. The fundamental distinctions we have found are related to the tools and the mathematical knowledge used by the students in each case.

The simulation and experimentation that took place in the modelling process had an influence on the students' production of meaning to their models and to the real situation.

Looking at these procedures on the modelling task from a semiotic point of view, the Peircean pragmatic perspective elucidates the influence of the *intention*, the *collateral experience* and the preview of *future consequences* on student meaning production.

In the present research, our focus was on the processes of experimentation and simulation and their relationship with the production of meaning in mathematical modelling activities. We analysed the processes developed by four groups of students from two different levels of education in one modelling task, considering Peirce's pragmatic perspective of meaning construction. The concepts of familiarity, intention, collateral experience and future consequences proved to be relevant for understanding the student's production of meaning in both cases of our study. However, those concepts are related to the sign production process itself (gestures, speeches, students' written records to express models and modelling) in a modelling task. This process – semiosis – could also be investigated in future research, preferably with a larger number of students. We also believe it will be useful to plan other studies that may involve different classes of students but also a larger number of modelling tasks, in order to understand what appears to be fundamental in the student's modelling processes in addition to the more immediate question of the mathematical background of the subjects.

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Mathematical models and meanings by school and university students in a modelling task

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Mathematical modelling can be beneficial for students at different levels of education. The present study endorses the self-differentiating character of modelling problems. It involves two classes from 9th grade in Portugal and an undergraduate mathematics class in Brazil. Students in both contexts were given a modelling task that required the development of a hand biometrics recognition system, during which they performed experimentation and simulation. The aims were to seek distinctions and commonalities between the models developed in both classes, and to know how simulation and experimentation influenced meaning production. We adopt a cognitive perspective of mathematical modelling, consistent with the modelling cycle as described in its several steps and sub-processes. A conceptualization of the so-called prototyping process was also presented through a schematic cycle, which typically involves the creation of a mathematical model and entails processes of experimentation and simulation. For a discussion on meaning production, we draw on Peirce's semiotic theory. Several of its concepts are promising for our analysis of student construction of meaning for the models produced and the modelling processes carried out. Thus, the concepts of familiarity, collateral experience, and identification of future consequences were examined and discussed from a pragmatic perspective on meaning production. The study follows a qualitative methodology with case study design in two classes. From each class we selected the work of two groups of students, which showed distinct models and ways of thinking on the problem. Our analysis then focused on their models and assumptions, and finally on meaning production for the models and modelling process. We conclude that there are many common aspects between the types of models developed by school and university students. They all took similar approaches to the real problem and had in mind the development of a prototype that should work properly. The prototype involved the construction of a biometric database and an algorithmic mathematical model. The main distinctions are related to the tools and the mathematical knowledge used by the students in each case. Moreover, experimentation and simulation were relevant elements for student meaning production, which is endorsed by a pragmatic perspective. The action of experimenting played a relevant role in understanding the real situation and its conditions, showing the students future consequences that gave meaning to the mathematical model; similarly, the simulation based on the mathematical model showed future consequences in the results generated by the model. This influence of experimentation and simulation is aligned with elements of pragmatism, according to Peirce's semiotic theory.