The object-tool duality in mathematical modelling. A framework to analyze students' appropriation of Sankey diagrams to model dynamic processes

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Abstract

Students often do not experience the relevance of learning mathematics. This paper reports on an exploratory case study, in which a class of grade 8 students (n=35) was introduced to Sankey diagrams. The aim was to explore to what extent these students could appropriate Sankey diagrams, meaning: they could describe these as objects in themselves and they could use them to model and visualize phenomena relevant to them. Based on Cultural-Historical Activity Theory, we developed an analytical construct defined as the object-tool duality, coordinating mathematics as a set of objects and as a set of tools. The analysis of students' answers showed that they could use these diagrams as tools to visualize phenomena. When asked to describe the object, all mentioned the tool-side. So, in their appropriation the tool-side came before the object-side. Our contribution is that teaching the tool-side of mathematics before the object-side may increase students' sense of the relevance of mathematics, which is a topic to develop for future research.

Keywords. Appropriation; mathematical modelling; object-tool duality; relevance (of mathematics); Sankey diagrams.

La dualidad objeto-herramienta en la modelización matemática. Un marco de análisis de la apropiación de los estudiantes de diagramas Sankey para modelar procesos dinámicos

Resumen

A menudo los estudiantes no perciben la relevancia de las matemáticas y de su aprendizaje. Este artículo presenta un análisis exploratorio de un caso de estudio, en el cual se introducen los diagramas de Sankey para estudiantes de octavo grado. El objetivo fue explorar hasta qué punto estos estudiantes se apropiaron de los diagramas Sankey, es decir, si los describían como objetos en sí mismos y si eran capaces de utilizarlos para modelizar y visualizar fenómenos relevantes para ellos. Con base en la Teoría Cultural-Histórica de la Actividad, desarrollamos un constructo analítico definido como la dualidad objeto-herramienta, que coordinan las matemáticas como conjunto de objetos y conjunto de herramientas. El análisis de respuestas de los estudiantes mostró cómo usaron estos diagramas a modo de herramientas para visualizar ciertos fenómenos. Cuando se les pidió que describieran el objeto, todos mencionaron los aspectos como herramienta. Así, en su proceso de apropiación la interpretación como herramienta se antepuso a la de objeto. Nuestra contribución es que el trabajo con la vertiente herramienta de las matemáticas antes que con la de objeto puede aumentar la relevancia que los estudiantes dan a las matemáticas, resultando ser una línea a desarrollar en futuras investigaciones.

Palabras clave. Apropiación; diagramas Sankey; dualidad objeto-herramienta; modelización matemática; relevancia (de la matemática).

1. Introduction

Within mathematics classrooms, some students ask about the relevance of what they are learning (Hernández-Martínez & Vos, 2018). Researchers observed that mathematics teachers then promise "it will come in useful", whilst simultaneously offering students artificial mathematical activities that neither connect to students' daily life experiences nor to future workplaces (Boaler, 2000; Gainsburg, 2008). The reasons given by mathematics teachers for not conveying relevance are lack of ideas, resources, and preparation time to search for good examples (Gainsburg, 2008). The teacher's message contrasts with the implicit message that mathematics is somehow important, since it is compulsory and prominent within the school timetable. Niss (1994) captured this situation as the *relevance paradox*, whereby the subjective irrelevance of mathematics as experienced by students in classrooms contrasts with its importance as expressed by teachers, parents, policy makers, and so forth.

To address the relevance paradox, there have been calls to better connect school mathematics to the world outside school by offering mathematical modelling tasks (Kaiser, 2014). These tasks start from real-life problems, for which mathematics is a tool needed in the process of answering them. Such tasks should assist students in *experiencing* the relevance of mathematics. However, many modelling tasks are paper- and pencil problems with constructed situations, ready-made formulas, and questions that would not have been asked by people in the situation (Vos, 2013, 2020). In fact, conveying the relevance of mathematics and connecting school mathematics to students' daily lives or to their future workplaces is not without problems. First, many mathematics teachers have little experience in designing tasks (1) that align with students' mathematical levels and interests, and (2) that offer an authentic context and have a question that people in that situation would pose (Vos, 2011). For example, in connecting industrial contexts to mathematics, Nilsen and Vegusdal (2017) observed teachers offering authentic figures from an enterprise, and then asking students to calculate the radius of a tank, or the surface area of a warehouse. In such tasks, the workplace is a context for a dressed-up word problem (Blum, 2015) and the answer neither solves a problem from the enterprise, nor assists a better understanding of industrial processes. Second, professionals in industries and governments perform contextualized activities like measuring, estimating error margins, and planning, which they often do not recognize as mathematical; Andreassen, Nilsen and Vos (submitted) report of workers in a sewage plant who stated to only use primary school mathematics (addition, multiplication and percentages) ignoring their constant administering and monitoring of measurements, reading off data from tables and graphs, working with uncertainty, relationships, and constantly interpreting the data in the situated context of the enterprise. These mathematical activities were hidden in routines and in instruments, and looked unrelated to school mathematics (Wake, 2014).

Thus, resolving the relevance paradox by connecting civic and industrial contexts to school-based mathematical activities is not straightforward, since both are rooted in different cultures, consist of different practices, and have different goals (FitzSimons & Boistrup, 2017; Frejd & Bergsten, 2016). On the one hand, there is the school culture, in which classroom activities are guided by short tasks detached from daily life, with the aim to prepare students for exams and more distantly, for later life. On the other hand, the out-of-school culture displays ongoing datafication with the aim to efficiently support administration, safety, customer service, planning and so forth. To access and operationalize the increasingly complex data, professionals are trained to apply routines, use instruments, and read and interpret the supply of data, which are increasingly displayed in colorful visualizations (Mayer-Schönberger & Cukier, 2013).

This paper draws on a research project that addresses the relevance paradox. We study how the relevance of mathematics for students' future lives can be conveyed, in particular for societal issues, such as recycling waste (Vos & Frejd, submitted), or sewage cleaning (Andreassen, Nilsen & Vos, submitted). The project aims to contribute to research on mathematical modelling education and the discussion whether modelling is taught to develop mathematics or as content in itself (Julie & Mudaly, 2007). We take a middle position is saying that neither mathematics nor mathematical modelling are aims in themselves, but that (1) both are useful in society and potentially in students' future lives, (2) students will be more motivated to learn mathematics when they understand its relevance, (3) this relevance can be conveyed through mathematical modelling experiences, and (4) students will be better at modelling in new situations when they already have rich modelling experiences.

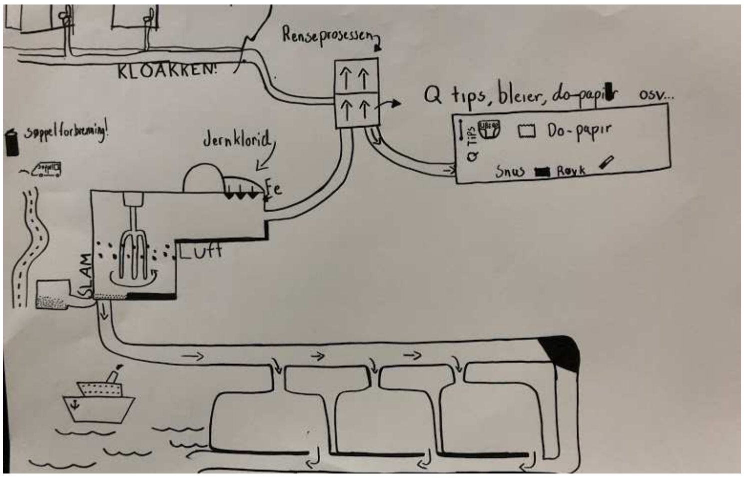


Figure 1. Grade 8 students' spontaneous diagram to describe a sewage process

We report from a lesson with students in 8th grade (13-14 years old) about the modelling of dynamic processes (demography, industries, etc.). The lesson built on a project where another group of 8th grade students had visited a sewage plant and sketched a diagram of the sewage process (Andreassen, Nilsen & Vos, submitted). Figure 1 shows this process: the excretion starts from the homes, goes through pipes, filters physical objects (e.g., toilet paper), it is chemically treated, mixed with air, separated from a sludge and safely released into the sea. After observing students' attempt to visualize the sewage process, we decided to offer another group of 8th grade students a tool: Sankey diagrams (explained below). We describe the lesson on Sankey diagrams, and how they appropriated them for use in situations that mattered to them.

2. Background and theoretical framework

2.1. Sankey diagrams

A Sankey diagram shows flows, of which the width represents the flow's quantity. Figure 2 shows a Sankey diagram of the population dynamics of Norway between 1978 and 2018. On the left a large vertical bar represents the 4.1 million inhabitants in 1978. From its top, there are two flows departing: 0.4 million inhabitants died and 0.2 million emigrated between 1978 and 1988. In this diagram, the widths of flows represent numbers of people, so the flow of 0.2 million deaths is twice as wide as the flow of 0.4 million emigrants. By comparing the width of the flows, one can see that the numbers of deaths and births hardly changed over the four decades after 1978, but the numbers for emigration and immigration did. In general, Sankey diagrams are *data visualizations* (i.e. data representations, statistical graphics) that can be used when all variables have the same unit (e.g. frequencies, liters, Euros, Joule). The name giver is Capt. Matthew Sankey, who around 1900 visualized energy loss in steam engines. However, forty years earlier, the French engineer Charles J. Minard already created

Sankey diagrams to visualize flows of people over time and space, like military movements in warfare, and cross-continental migration (Tufte, 2001).

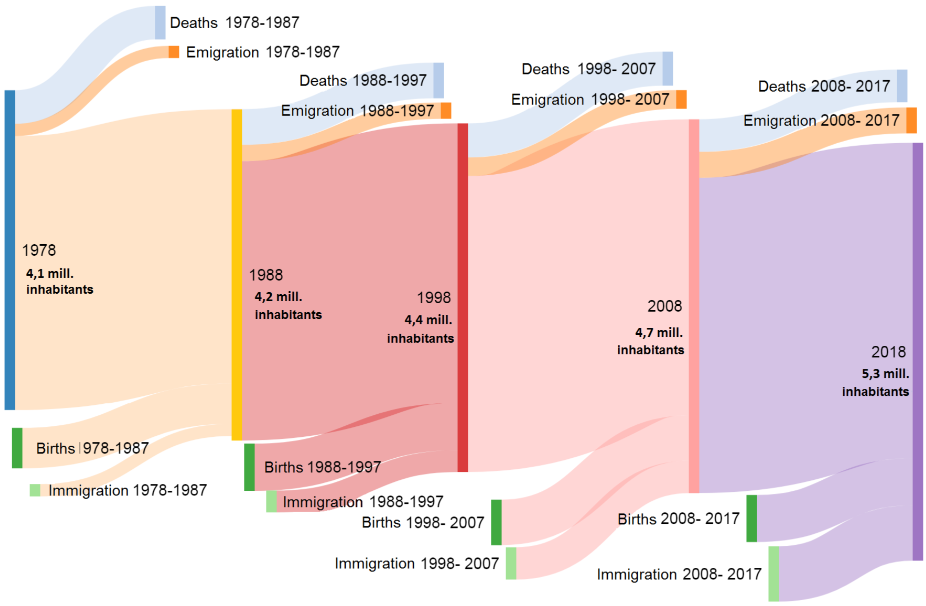


Figure 2. Norway's population dynamics 1978-2018 visualized through a Sankey diagram

Data visualized in a pie chart can be visualized in a Sankey diagram, but not vice-versa. A pie chart shows a whole and its parts, which can be displayed by a Sankey diagram as a flow dividing into sub-flows. A Sankey diagram has the additional option to visualize sub-flows of sub-flows, and the diverse flows and sub-flows then can be woven and make a new whole. In this way, they can represent a process, for example how magnitudes change over time (Figure 2) or place. The sewage process in Figure 1 can be redrawn with flows, of which the width represents the sewage fluid quantities. Sankey diagrams are also used to visualize complex data sets, for example how voters move from one party to another between elections (British Election Study, 2019).

Sankey diagrams have gained popularity, driven by a growing datafication in society (Mayer-Schönberger & Cukier, 2013), and media becoming more visual. These diagrams cannot be created with standard spreadsheet programs, but there are simple, free, web-based tools, like *SankeyMATIC*, allowing to code each flow through its starting point, quantity and end point. The first codes for Figure 2 were:

1978 [424697] Deaths 1978-1987; 1978 [155351] Emigration 1978-1987; 1978 [3471160] 1988; Births 1978-1987 [514207] 1988; Immigration 1978-1987 [155351] 1988; 1988 [450515] Deaths 1988-1997; 1988 [205481] Emigration 1988-1997; ….

Sankey diagrams, just like all data visualizations, are tools of a mathematical nature to visualize, and thereby model, large or complex data. They can be used to describe phenomena from real life, and eventually analyze or solve problems. Thus, a Sankey diagram is a mathematical model, in which quantitative information is structured visually through colorful nodes and flows with two goals: to analyze data (for oneself to understand) and to communicate these (for others to understand).

2.2. Theoretical stance and the *object-tool duality* in mathematical modelling

In our study, we use Cultural-Historical Activity Theory (CHAT) to analyze students' work with Sankey diagrams. CHAT builds on Vygotsky (1978) who developed psychological research with the premise that people and their minds cannot be studied detached from the activities and the socio-cultural environment. In CHAT, an activity is framed as a triad consisting of a *subject* (a person), an *object* (what the activity is about), and the cultural *tools* used (language, gestures, physical objects, etc.), see Figure 3 (top). An example of such a triad is a student (subject) studying the derivative (object) using textbooks, a digital graph plotter, symbols and verbal language (tools), see Figure 3 (bottom, left). According to CHAT, all tools are cultural and historical, since they originate from a wider world beyond the activity and were developed by people who, most likely, aren't involved in the present activity (Roth & Radford, 2011). In the example of the student learning the derivative, the symbols, the textbooks and the graph plotter are tools that were developed by people external to the classroom and taken into use by others (cultural), and there is a time lapse between the development of the tools and the present learning activity (historical).

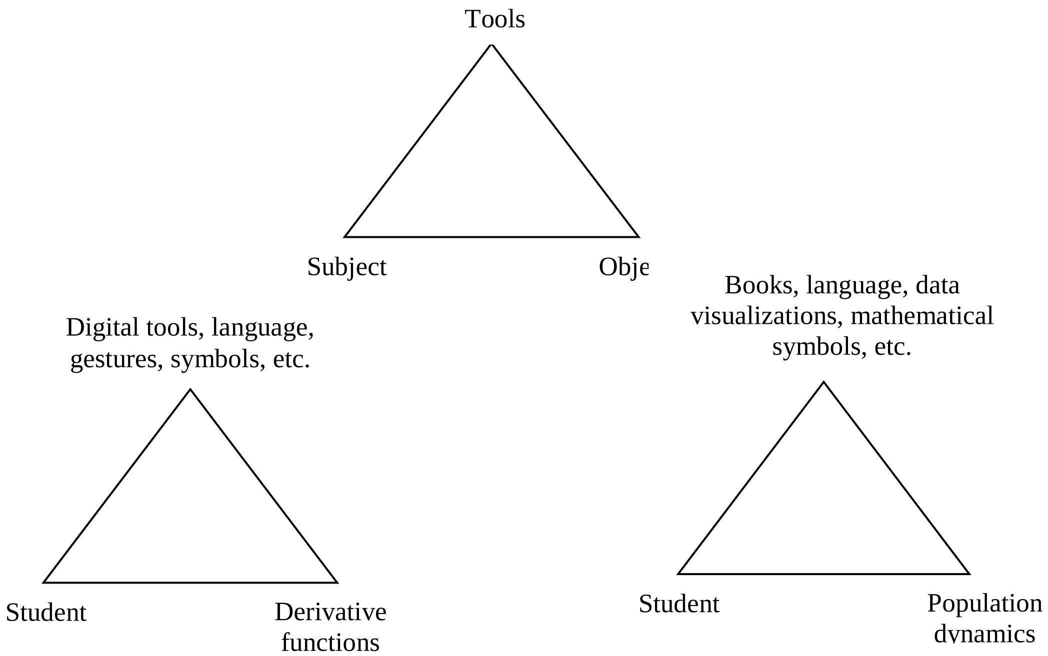


Figure 3. Vygotsky's Mediational Triangle (top) with two concretizations

An important element in CHAT is that the subject's activity is *mediated* (guided, facilitated, and modified) by the tools used in the activity. For example, the digital graph plotter allows for varying graphs of functions dynamically. Before such plotters became available to students, the learning activities were based on static graphs of functions in a textbook, which made the learning different in many ways.

Since CHAT offers a frame to analyze how cultural tools mediate activities, it is a helpful theory for researchers of mathematics education analyzing teaching and learning activities and the mediation thereof by, for example, signs and gestures (Radford, 2010), or by digital tools (Monaghan, 2016; Williams & Goos, 2012). When researching mathematical modelling education, the CHAT triad can yet again be used, but now, the position of mathematics changes. It is no longer the object, but one of the tools to solve real-life problems. For example, when a student works on a task about population dynamics and uses a Sankey diagram to model these, the diagram is not the object of the activity, but the tool used in the activity (Figure 3, bottom right). The combination of, and the difference between, mathematics as a set of objects and as a set of tools is what we indicate by the *object-tool duality* in mathematical modelling.

To see mathematics primarily as a set of objects is often the perspective of mathematics teachers. As described in the introduction, many teach the subject as aim in itself, disconnected from the tool-side. In fact, when people encounter mathematical objects, such as derivatives, for the first time in their lives, these objects are neither presented nor experienced as tools. This leads students to ask for the relevance of what they learn, which points at their need to understand the tool-side of the object.

To see mathematics as a set of tools is the perspective of users, like the biological or psychological researchers who click some buttons in a statistical package to obtain a *p-value* for their experimental results. Also, this perspective is the one of the people in history, who developed mathematical concepts because they needed these. In the 17th century, Newton and others developed the derivative as a tool to study gravity and astronomy (Boyer, 1959). Minard developed the Sankey diagram to visualize flows of large groups. Both needed mathematical tools to model non-mathematical phenomena.

Using CHAT as a lens, we perceive Sankey diagrams as both mathematical objects in themselves, since they can be defined as a class of data visualizations decontextualized from their use. Simultaneously, we perceive them as cultural-historical tools that proved useful in modelling phenomena. Following CHAT, tools mediate activities, which applies to mathematical tools as well. Whether one uses tables or Sankey diagrams will affect the people studying, for example, population dynamics. The mathematical tools impact their insights into relationships, their trust in the analysis, their joy in the activity, or their status when showing their work to others.

In mathematical modelling, when people use mathematics as a set of tools, they most likely *consciously* use these, and hence the use of mathematical tools differs from how we unconsciously use language, and how some people unconsciously use pens or smart phones. To use mathematical tools requires learning. After having encountered them, their use requires reflection, for example on how to represent the object (algebraically, graphically, etc.), or how to adapt the object to specific activity. The mathematical tool is thus an object of learning and, while in use, an object of reflection, which justifies speaking of the *object-tool duality* in mathematical modelling. This duality sustains our analysis of Sankey diagrams as objects of learning in schools and as tools to model and visualize phenomena. CHAT enables us to capture both the object-side and the tool-side of the diagrams. This *object-tool duality* in mathematical modelling frames the analysis in our research as an analytic construct.

The *object-tool duality* offers a perspective on mathematical objects like division algorithms, percentages, or Pythagoras' Theorem, indicating that each of these objects also has a tool-side, which shows when these are used in mathematical modelling. The *object-tool duality* as a lens on (pure) mathematics education opens a window on analyzing how mathematical objects are taught for the sake of an abstract curriculum, as objects of learning without considering their history, the rationale for humans to develop them, and their usefulness in non-mathematical activities in real life beyond the school. The *object-tool duality* then reveals how mathematical objects lack connections by being stripped of their tool-side, which leads to the relevance questions asked by students, as described in the introduction of this paper.

2.3.1 Appropriating mathematical object-tools

Following educational researchers who applied CHAT, we speak of learning as *appropriation* (Moschkovich, 2004; Radford, 2010; Säljö, 1991). Appropriation is the process of making something one's 'property'. Radford (2010) says: "we do not mean that students' knowledge appropriation is achieved through a kind of crude transfer of information coming from the teacher. As we see it, knowledge appropriation is achieved through the tension between the students' subjectivity and the social means of semiotic objectification" (p.241). Vos and Roorda (2018) explain that the process of appropriation is hard to capture by a researcher, but the result of a successful appropriation can be observed: it is when a student has ownership over a mathematical object as a personalized tool, when (s)he can use it flexibly, confidently and strategically in a variety of non-routine situations, without being prompted. In this definition, one can recognize the *object-tool duality* described before. A successful appropriation implies that a student knows what the object is (factual knowledge), how it can be used (procedures, algorithms), can make connections to different situations and with other objects (conceptual knowledge), and has knowledge of it as a tool: of its aims, its limitations, and when/where/for what purpose it can be used. In case of Sankey diagrams, this means that students understand how they are constructed as tools for visualizing phenomena with nodes and flows, recognize that the width represents the flow's quantity, can read off data, can explain how they differ from other visualizations (e.g. pie charts), and can use them creatively in new situations.

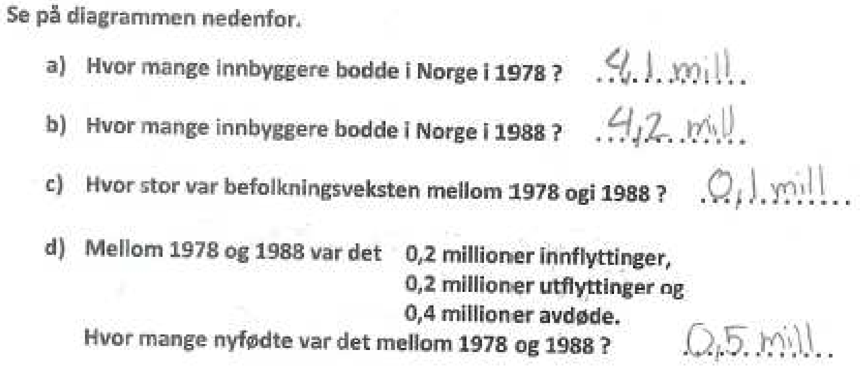
For researching learning processes in mathematical modelling education, the appropriation of mathematical object-tools is as an analytic construct that addresses that mathematical tools for solving real life problems still have an object side. This means that students need opportunities to encounter and learn about these cultural tools, that is, how these mathematical tools are constructed and used by others. In this appropriation process, the tools are objects of learning, which subsequently and hopefully, become personalized tools. In the present paper, the *object-tool duality* and students' appropriation are central tenets. The research question was: *to what extent can grade-8 students appropriate Sankey diagrams as object-tools, so they can describe them as objects in themselves and use them for situations that matter to them?*

3. Methods

We introduced Sankey diagrams to a class of 35 grade 8 students (13-14 years old) from an urban school in Norway within a mathematics lesson of 90 minutes. The class was split in two halves (n=17 and n=18) and their lesson could be scheduled within the same week, so all students were at the same level of learning. The first author was the teacher, whilst the regular mathematics teacher remained in the background. The aim was to offer students a variety of Sankey diagrams so they could gradually generalize across contexts, describe features of the diagrams, and use them in new situations.

The lesson was based on a booklet with tasks (Vos, 2019) described below. The lesson had a short introduction (5 minutes), in which the teacher introduced herself and how the students were to work with the booklet. She did inform the students of the topic of the lesson, but no Sankey diagrams were shown or explained at that stage. The students sat in groups of 2 or 3, making 15 groups in total. The groups received jointly one booklet, so students alternated their reading of texts and their writing of answers into the booklet. Both teachers were available for consults, sometimes helping to interpret tasks. However, they were often chatting with each other, since the groups mostly worked independently. After 40 minutes there was a 5-minute break, in which the students left the classroom for the playground. In the part after the break, all student groups reached the final page of the booklet. In the final 5 minutes, the teacher offered a summary, thanked the group, and collected the booklets.

The booklet consisted of seven pages and covered four topics: Norway's population dynamics 1978-2018, the milk factory, grade 10 students' choices for theoretical or vocational tracks, and energy conversion in light bulbs. Each topic centered on a Sankey diagram visualizing a phenomenon, about which several tasks were to be completed. For each topic, the tasks increased in complexity and were concluded by a modelling problem, such as *what are reasons for Norway's population growth?* Or, *what are differences in energy usage between classical light bulbs and energy saving bulbs?* To frame the increasing complexity, the task design was based on the levels of reading, interpreting and creating mathematical graphics from Friel, Curcio and Bright (2001) and Prodromou (2017): (1) reading data *in* the diagram, which entails looking at local features in the diagram, (2) 'reading *between*' data in the diagram, which entails examining relationships between variables, or between variables and the described phenomena, (3) 'reading *beyond'* the diagram, which entails analyzing the diagram from a meta-perspective, for example by asking "what if..?" questions (e.g. what if one variable increases?), and by critiquing ("if the diagram shows flows of milk, yoghurt and ice cream, it will not show the financial profit"), and (4) creating diagrams, which entails calculating widths for the flows to be drawn.


Se på diagrammen nedenfor.

a) Hvor mange innbyggere bodde i Norge i 1978 ? 4,1 mill

b) Hvor mange innbyggere bodde i Norge i 1988 ? 4,2 mill

c) Hvor stor var befolkningsveksten mellom 1978 ogi 1988 ? 0,1 mill

d) Mellom 1978 og 1988 var det 0,2 millioner innflyttinger, 0,2 millioner utflyttinger og 0,4 millioner avdøde.
Hvor mange nyfødte var det mellom 1978 og 1988 ? 0,5 mill

Figure 4. Tasks on Norway's population dynamics (for translation, see the text below)

We illustrate these levels through the first tasks in the booklet, see Figure 4. These tasks are about the population dynamics in Figure 2. Tasks a) and b) ask how many inhabitants lived in Norway in 1978 and 1988. So, students had to read *in* the diagram. Task c) ask to calculate the growth in this decade. The answer cannot be read in the diagram, requires a calculation, and this is termed as 'reading *between*' data in the diagram. Task d) offer numbers for migration and deaths, and asked students to find the number of births between 1978 and 1988. This task is, yet again, a 'reading *between*'. Only later-on in the booklet came tasks of a higher level, for example in comparing a Sankey diagram to a pie chart. The tasks to create a Sankey diagram, requiring calculations of the widths of flows were scaffolded as follows. There was no creating task with the first Sankey diagram. With the second, the students were offered a Sankey diagram, in which on the left-hand side, the cow and goat milk flowed into the factory, and on the right-hand side, the milk cartons, cheese, yoghurt and ice cream exited the factory. Here, students had to reason about widths of flows, and to calculate how many liters of milk were represented by a flow drawn 1 cm wide. With the third diagram, students were to draw sub-flows for girls and boys within the flows of students choosing health studies and electronics. This, again, required them to calculate widths. In the fourth topic, on the conversion of electricity into light and heat, there was a given Sankey diagram about a classical lightbulb, and students were asked to draw one for the energy saving bulb. They were asked to calculate widths of flows, with the hint: “Take 1 Joule = 1 mm”. This was the first occasion, where a modelling problem (*what are differences in energy usage between classical light bulbs and energy saving bulbs?*) required students to draw their own Sankey diagram.

We assumed that with four different Sankey diagrams, students would be able to find common characteristics. The tasks on calculating widths aimed at raising awareness on the quantitative nature of the diagrams. Throughout the booklet, there was no definition of Sankey diagrams given, since this would guide students when being asked, on the final page, what a Sankey diagram is (object-side) and to draw a Sankey diagram visualizing an issue of their own choice (tool-side). This final page was designed as main data source for analyzing students’ appropriation of Sankey diagrams. At the end of the lesson, we collected the 15 booklets with students’ answers. We supplemented these with observation notes made during the lesson, and a report written after the lesson. We didn’t film or audiotape the students, not wanting to disturb them. We analyzed the students’ written answers by evaluating to what extent they showed appropriation of Sankey diagrams as a tool and a as an object.

4. Results

The first tasks in the booklet concerned the population dynamics in Norway between 1978 and 1988, see Figure 4. The answers show that most students were well able to interpret the data in the diagram and meaningfully make calculations with these. One of the observers noted that a student explained her peers that immigrants were to be “plussed” and emigrants and deaths were to be “minussed” from the total. This illustrates how the students used mathematical operations as a tool in the tasks. All but one student group found correct answers, which shows that many started to appropriate the diagram as a visualization tool showing data of population dynamics. One group interpreted question d), see Figure 4, as a task to perform an addition, since it showed three numbers written vertically; they wrote 0,8 (= 0,2 + 0,2 + 0,4) as answer. These students perceived this task as a mathematical object, detached from it being a tool to meaningfully describe population dynamics. However, on the ensuing tasks in the booklet, this group picked up in understanding the diagram as a tool for visualizing population dynamics. In the end, all students groups were able to use the Sankey diagram in Figure 2 as a tool for giving satisfactory answers to the modelling problem: *what are reasons for Norway’s population growth?*

In the next tasks regarding the Sankey diagram of the milk factory, the students were asked to reason about the widths of the different flows (of milk, yoghurt, etc.). In comparing the widths of flows, they calculated widths, and some used their ruler to rather measure these. They wrote sentences like “when the liters are more, the bar is thicker” or “when the goat milk is half the ecological milk, the arrow is half as big”. Here, the flows were objects of reasoning, and also tools for representing milk quantities. Thereafter, the students encountered a Sankey diagram showing the distribution of students in senior secondary and their numbers in the different vocational and theoretical tracks. Here, students analyzed what tracks were mostly chosen by girls (health) and boys (electronics) and drew these as flows. All groups did well in using the width of the flows as a tool to visualize the different quantities. Thereafter came the tasks about the conversion of electrical energy in light bulbs. Here we saw groups correctly use the hint “Take 1 Joule = 1 mm” and draw the flow of 20 Joule with a width of 2 cm. This enabled them to visualize the energy conversion in an energy saving bulb and explain differences with a classical bulb. So, they were able to create a new Sankey diagram as an object requiring mathematical calculations, and as a tool to visualize energy flows. The remaining groups drew qualitative flow charts. In the observation notes, it was written that this task was considered less interesting and that when seeing others advancing to the next tasks, they wanted to catch up.

On the last page of the booklet, the students were asked: what is a Sankey diagram, and what it can be used for? Typical student answers were:

“*It now shows, for example, what parts a thing goes into. Statistics*”

“*You can use it to show what goes in and what goes out*.”

“*A diagram that shows development*”

Many answers contained the word ‘it’, indicating that the students, to a certain extent, recognized commonalities across the four Sankey diagrams they had seen. These diagrams together had become one object for them. Also, most answers contained the word ‘show’ indicating that the students recognized that Sankey diagrams could be used purposefully as a tool to visualize phenomena from real life.

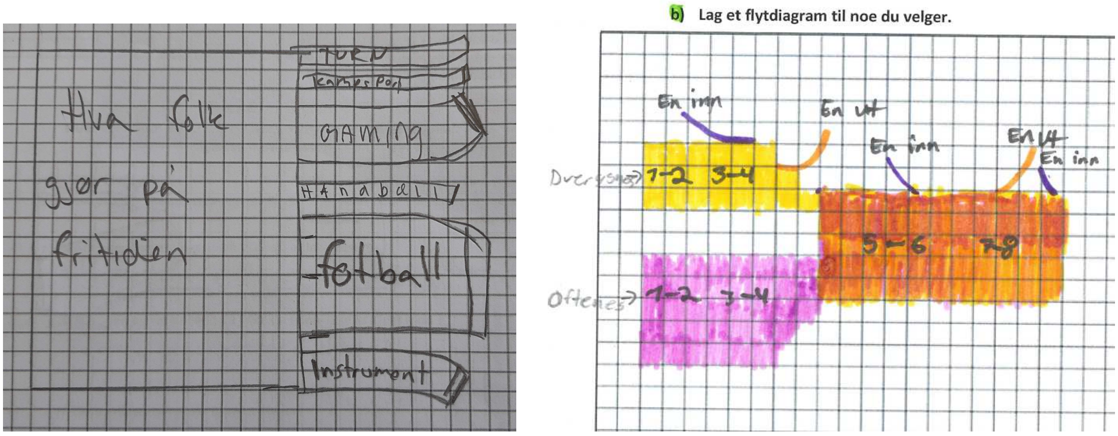


Figure 5. Two different Sankey diagrams creaated by the students

The final question in the booklet asked students to create a Sankey diagram on an issue of their interest. All student groups engaged in this enthusiastically. Most created diagrams like Figure 5 left, showing a whole and its parts, for example visualizing “what people do in their free time” or “how I spend my pocket money”. The data visualized could also have been shown through a circle diagram. These diagrams were probably inspired by the third Sankey diagram in the booklet, displaying grade 10 students’ choices for vocational or theoretical tracks. A different type of Sankey diagram was created by two groups, one in each of the half classes. It showed the population dynamics of how their class had evolved historically. One group showed it for the past four years, the other for the past eight years (Figure 5, right). The diagram showed that the students had come from two different primary schools, that some individuals had joined, and some had left. In this diagram, parts became a whole, and further developed over time into a new whole, showing a dynamic process that one cannot display through the standard data visualizations available in standard spreadsheet programs. Most likely, such a class population diagram was inspired by the first Sankey diagram in the booklet (Figure 2). In all self-produced Sankey diagrams, the students had drawn the flows that visualized smaller quantities clearly thinner, but in none they drew the width of flows proportionally to their quantity. Not one group made an effort to calculate accurately the width of flows in their self-designed Sankey diagrams. Despite the tasks on calculating widths, students’ own

Sankey diagrams were qualitative tools, similar to the somewhat naive diagram that their peers had created earlier visualizing the sewage process (Figure 1).

When we analyze students' appropriation of Sankey diagrams in light of the object-tool duality, we can see, on the one hand, that students understood them as objects, since they were able to describe them and knew what to do when asked to create one. However, the characteristic regarding the proportionality of the widths of flows hadn't been appropriated, neither in their descriptions nor in their drawings. They primarily saw them as qualitative objects. On the other hand, the students understood well that Sankey diagrams could be tools to visualize data, and they could use them creatively to represent situations different from the ones presented to them, such as for their sports, their pocket money or for their own class population dynamics. So, they could use them to visualize issues that mattered to them.

5. Conclusion

In the present study, we carried out an exploratory classroom experiment guided by the research question: *to what extent can grade-8 students appropriate Sankey diagrams as object-tools, so they can describe them as objects in themselves and use them for situations that matter to them?* Some students told us they had not seen such diagrams before, and they engaged with them for only 90 minutes. We observed them appropriating the Sankey diagrams as objects to a certain extent: they started to use the name, they were able to indicate commonalities across the different Sankey diagrams; they could describe some characteristics. Since we did not ask them to compare with other tools (e.g., pie charts), we do not know whether they could do so. We observed the students disregarded that the widths of the flows really mattered; they seemed to take the flows more qualitatively. We also looked at critical aspects of students' appropriation of Sankey diagrams as tools; we did not see them use these flexibly and confidently without being prompted. That would be a too high expectation for 8^{th} grade students after having encountered four Sankey diagrams and being prompted to draw a fifth. However, all were well able to describe the tool-side of the diagram, as expressed in sentences like "a diagram that shows development". They understood that the Sankey diagrams could be useful for visualizing and modelling phenomena.

Our study indicates that students can learn about the tool-side of a mathematical object *before* fully grasping the object-side and without having seen a definition. However, this indication should be seen in relation to the design of the 90 minutes explorative classroom study, the sample of students and the design of the booklet. It is therefore not possible to generalize our findings, yet these confirm results from previous research showing that one does not need to teach mathematical concepts as isolated objects before teaching applications. This approach, known as *applicationism*, wrongly assumes a deductive order in teaching the abstract before the use (Barquero, Bosch & Gascón, 2013). Second, our study shows that students can gain a sense of the relevance of mathematics, without fully grasping the underlying mathematics; they can comprehend that the tool-side of mathematics can assist in visualizing phenomena, and that this assists in better understanding such phenomena. Third, when students learn about the tool-side in conjunction with, or even before the object-side, many of them will be more motivated to learn the abstract object-side, since they start from an answer to their relevance question. They will then better see a purpose in their learning efforts, knowing they will obtain a tool for solving problems on issues that matter to them. This will mitigate the relevance paradox in mathematics (Niss, 1994).

By appropriating Sankey diagrams, the students in our study experienced a certain relevance of mathematics (Hernández-Martínez & Vos, 2018), on the one hand by using these diagrams to visualize phenomena and understand these for themselves, but also to the tool for communicating with others. We observe potential areas for further studies, namely regarding the use of mathematical models not only to solve real-life problems, but also to use these as tools for communication with those whose problems are addressed. Also, research could reach beyond Sankey diagrams, exploring the use of computational tools, simulations, or animated data visualizations, what role these can play in mathematical modelling education, what position these could have in the curriculum, how these can play a role in inter-disciplinary education, and so forth.

Finally, the study presented shows how the *object-tool duality* is a simple, yet powerful construct for analyzing mathematical modelling activities. By discerning between the more abstract object-side and the applicable tool-side, we do justice to how humans historically developed mathematical objects for their usefulness to model and visualize non-mathematical phenomena. The *object-tool duality* acknowledges the abstract side of mathematics, whilst simultaneously including its concrete usefulness. Tensions between both sides are not easily resolved, and we recommend further research into educational settings that build on the *object-tool duality* of mathematics.

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The object-tool duality in mathematical modelling. A framework to analyze students' appropriation of Sankey diagrams to model dynamic processes

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Students often do not experience the relevance of learning mathematics. To address this relevance problem, we run a mathematical modelling project, in which we create links between school mathematics and enterprises in the region. In an earlier project, we had observed that students needed a tool to describe dynamic processes. This paper reports on an exploratory case study, in which a class of grade-8 students was introduced to Sankey diagrams, a type of flow diagram to model and visualize dynamic processes. In Sankey diagrams, the width of a flow is drawn proportionally to the quantity depicted. The aim of the study was to explore to what extent the students could appropriate Sankey diagrams, meaning: they could describe these as objects in themselves and they could use them to model and visualize dynamic phenomena relevant to them. Based on Cultural-Historical Activity Theory, we developed an analytical construct defined as the *object-tool duality*, coordinating mathematics as a set of objects and as a set of tools. The object-tool duality has a historical base, since mathematical concepts, theorems or algorithms were developed by people, who needed these mathematical objects to solve non-mathematical problems, like in physics, astronomy, demography, and so forth. The object-tool duality emphasizes that mathematical objects have a tool-side. Detaching mathematical objects from their usefulness, as it is done in traditional mathematics teaching, explains students' lack of relevance experiences in mathematics classrooms. The empirical base of our study was a lesson of 90 minutes, in which students worked in small groups on tasks featuring Sankey diagrams. The diagrams depicted, for example, the population growth of Norway in the past 40 years, and another one showed the industrial process, in which milk becomes yoghurt or ice cream. Students worked on *what are reasons for Norway's population growth?* Or *what are differences in energy usage between classical light bulbs and energy saving bulbs?* They did ample tasks calculating the widths of the flows proportionally to the quantities. At the end of the lesson, students were asked to describe Sankey diagrams (object-side) and to draw one about an issue of their interest (tool-side). The analysis of students' answers showed that they could use these diagrams as tools to visualize phenomena, but they did this qualitatively. In their descriptions of the object, they used the name and mentioned the visualizing property, that is, the tool-side. None of them mentioned the proportionality of the widths of flows despite having done exercises on this. The lesson was too short, and the students were too few to enable us to draw general conclusions, but we it seems that in students' appropriation the tool-side was prominent. Our contribution is that teaching the tool-side of mathematics before the object-side may increase students' sense of the relevance of mathematics, which is a topic to develop for future research.